Controllability and state transfer

- state transfer
- reachable set, controllability matrix
- minimum norm inputs
- infinite-horizon minimum norm transfer

State transfer

consider $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$

we say input $u : [t_i, t_f] \to \mathbb{R}^m$ *steers* or *transfers* state from $x(t_i)$ to $x(t_f)$ (over time interval $[t_i, t_f]$)

(subscripts stand for *initial* and *final*)

questions:

- where can $x(t_i)$ be transferred to at $t = t_f$?
- how quickly can $x(t_i)$ be transferred to some $x_{\mathtt{target}}$?
- how do we find a u that transfers $x(t_i)$ to $x(t_f)$?
- how do we find a 'small' or 'efficient' u that transfers $x(t_i)$ to $x(t_f)$?

Reachability

consider state transfer from $x(0) = 0$ to $x(t)$

we say $x(t)$ is *reachable* (in t seconds or epochs)

we define $\mathcal{R}_t \subseteq \mathbb{R}^n$ as the set of points reachable in t seconds or epochs

for CT system $\dot{x} = Ax + Bu$,

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau \;\middle|\; u : [0, t] \to \mathbb{R}^m \right\}$$

and for DT system $x(t+1) = Ax(t) + Bu(t)$,

$$\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \;\middle|\; u(0), \ldots, u(t-1) \in \mathbb{R}^m \right\}$$

Reachable set

- $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$
- $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$
  ($i.e.$, can reach more points given more time)

we define the *reachable set* $\mathcal{R}$ as the set of points reachable for some t:

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$

Cayley-Hamilton theorem

if $p(s) = a_0 + a_1 s + \cdots + a_k s^k$ is a polynomial and $A \in \mathbb{R}^{n\times n}$, we define

$$p(A) = a_0 I + a_1 A + \cdots + a_k A^k$$

Cayley-Hamilton theorem: for any $A \in \mathbb{R}^{n\times n}$ we have $\mathcal{X}(A) = 0$, where $\mathcal{X}(s) = \mathbf{det}(sI - A)$

example: with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have $\mathcal{X}(s) = s^2 - 5s - 2$, so

$$\begin{aligned} \mathcal{X}(A) &= A^2 - 5A - 2I \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I \\ &= 0 \end{aligned}$$

Reachability for discrete-time LDS

DT system $x(t+1) = Ax(t) + Bu(t)$, $x(t) \in \mathbb{R}^n$

$$x(t) = \mathcal{C}_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

where $\mathcal{C}_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$ so reachable set at t is $\mathcal{R}_t = \textbf{range}(\mathcal{C}_t)$

by Cayley-Hamilton theorem, we can express each A^k for $k \geq n$ as linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, $\textbf{range}(\mathcal{C}_t) = \textbf{range}(\mathcal{C}_n)$

thus we have

$$\mathcal{R}_t = \begin{cases} \textbf{range}(\mathcal{C}_t) & t < n \\ \textbf{range}(\mathcal{C}) & t \geq n \end{cases}$$

where $\mathcal{C} = \mathcal{C}_n$ is called the *controllability matrix*

- any state that can be reached can be reached by $t = n$
- the reachable set is $\mathcal{R} = \textbf{range}(\mathcal{C})$

Controllable system

system is called *reachable* or *controllable* if all states are reachable ($i.e.$, $\mathcal{R} = \mathbb{R}^n$)

system is reachable if and only if $\textbf{rank}(\mathcal{C}) = n$

example: $x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is $\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \textbf{range}(\mathcal{C}) = \{ \ x \mid x_1 = x_2 \ \}$$

General state transfer

with $t_f > t_i$,

$$x(t_f) = A^{t_f - t_i} x(t_i) + \mathcal{C}_{t_f - t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

hence can transfer $x(t_i)$ to $x(t_f) = x_{\mathrm{des}}$

$$\Leftrightarrow \qquad x_{\mathrm{des}} - A^{t_f - t_i} x(t_i) \in \mathcal{R}_{t_f - t_i}$$

- general state transfer reduces to reachability problem
- if system is controllable any state transfer can be achieved in $\leq n$ steps
- important special case: driving state to zero (sometimes called regulating or controlling state)

Least-norm input for reachability

assume system is reachable, $\textbf{rank}(\mathcal{C}_t) = n$

to steer $x(0) = 0$ to $x(t) = x_{\mathrm{des}}$, inputs $u(0), \ldots, u(t-1)$ must satisfy

$$x_{\mathrm{des}} = \mathcal{C}_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

among all u that steer $x(0) = 0$ to $x(t) = x_{\mathrm{des}}$, the one that minimizes $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$ is given by

$$\begin{bmatrix} u_{\mathrm{ln}}(t-1) \\ \vdots \\ u_{\mathrm{ln}}(0) \end{bmatrix} = \mathcal{C}_t^T (\mathcal{C}_t \mathcal{C}_t^T)^{-1} x_{\mathrm{des}}$$

u_{ln} is called *least-norm* or *minimum energy* input that effects state transfer

can express as

$$u_{\mathrm{ln}}(\tau) = B^T (A^T)^{(t-1-\tau)} \left(\sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\mathrm{des}},$$

for $\tau = 0, \ldots, t-1$

Minimum energy

$\mathcal{E}_{\min}$, the minimum value of $\displaystyle\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$ required to reach $x(t) = x_{\text{des}}$, is sometimes called *minimum energy* required to reach $x(t) = x_{\text{des}}$

$$
\begin{aligned}
\mathcal{E}_{\min} = \sum_{\tau=0}^{t-1} \|u_{\text{ln}}(\tau)\|^2 &= \left(\mathcal{C}_t^{\mathsf{T}}(\mathcal{C}_t\mathcal{C}_t^{\mathsf{T}})^{-1}x_{\text{des}}\right)^{\mathsf{T}} \mathcal{C}_t^{\mathsf{T}}(\mathcal{C}_t\mathcal{C}_t^{\mathsf{T}})^{-1}x_{\text{des}} \\
&= x_{\text{des}}^{\mathsf{T}}(\mathcal{C}_t\mathcal{C}_t^{\mathsf{T}})^{-1}x_{\text{des}} \\
&= x_{\text{des}}^{\mathsf{T}}\left(\sum_{\tau=0}^{t-1} A^{\tau}BB^{\mathsf{T}}(A^{\mathsf{T}})^{\tau}\right)^{-1} x_{\text{des}}
\end{aligned}
$$

- $\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (*i.e.*, how large a u is required)
- $\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of x_{des}, t)
- ellipsoid $\{ z \mid \mathcal{E}_{\min}(z, t) \le 1 \}$ shows points in state space reachable at t with one unit of energy (shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)

Energy dependence on time

$\mathcal{E}_{\mathrm{min}}$ as function of t:

if $t \geq s$ then

$$\sum_{\tau=0}^{t-1} A^\tau BB^\mathsf{T}(A^\mathsf{T})^\tau \geq \sum_{\tau=0}^{s-1} A^\tau BB^\mathsf{T}(A^\mathsf{T})^\tau$$

hence

$$\left(\sum_{\tau=0}^{t-1} A^\tau BB^\mathsf{T}(A^\mathsf{T})^\tau\right)^{-1} \leq \left(\sum_{\tau=0}^{s-1} A^\tau BB^\mathsf{T}(A^\mathsf{T})^\tau\right)^{-1}$$

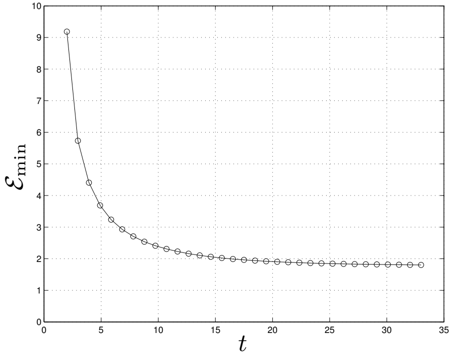
so $\mathcal{E}_{\mathrm{min}}(x_{\mathrm{des}}, t) \leq \mathcal{E}_{\mathrm{min}}(x_{\mathrm{des}}, s)$

$i.e.$: takes less energy to get somewhere more leisurely

Example:

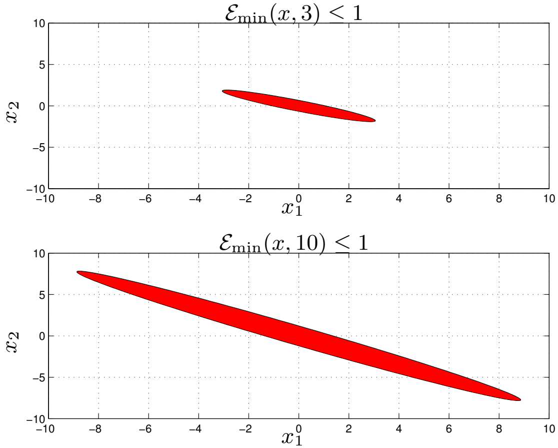
$$x(t+1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$\mathcal{E}_{\min}(z,t)$ for $z = [1\ 1]^T$:



Example

ellipsoids $\mathcal{E}_{\min} \leq 1$ for $t = 3$ and $t = 10$:



Minimum energy over infinite horizon

the matrix

$$P = \lim_{t \to \infty} \left(\sum_{\tau=0}^{t-1} A^\tau B B^\mathsf{T} (A^\mathsf{T})^\tau \right)^{-1}$$

always exists, and gives the minimum energy required to reach a point x_{des} (with no limit on t):

$$\min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \;\middle|\; x(0) = 0,\ x(t) = x_{\mathrm{des}} \right\} = x_{\mathrm{des}}^\mathsf{T} P x_{\mathrm{des}}$$

if A is stable, $P > 0$ ($i.e.$, can't get anywhere for free)

if A is not stable, then P can have nonzero nullspace

- $Pz = 0$, $z \neq 0$ means can get to z using u's with energy as small as you like

  (u just gives a little kick to the state; the instability carries it out to z efficiently)

- basis of highly maneuverable, unstable aircraft

Continuous-time reachability

consider now $\dot{x} = Ax + Bu$ with $x(t) \in \mathbb{R}^n$

reachable set at time t is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau \;\middle|\; u : [0, t] \to \mathbb{R}^m \right\}$$

fact: for $t > 0$, $\mathcal{R}_t = \mathcal{R} = \textbf{range}(\mathcal{C})$, where

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is the controllability matrix of (A, B)

- same $\mathcal{R}$ as discrete-time system
- for continuous-time system, any reachable point can be reached as fast as you like (with large enough u)

Proof, one direction only

we show for *any* u (and $x(0)=0$) we have $x(t) \in \textbf{range}(\mathcal{C})$. Write e^{tA} as power series:

$$e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots$$

by C-H, express $A^n, A^{n+1}, \ldots$ in terms of $A^0, \ldots, A^{n-1}$ and collect powers of A:

$$e^{tA} = \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha_{n-1}(t)A^{n-1}$$

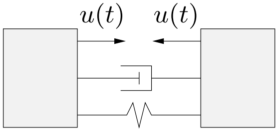
therefore

$$\begin{aligned} x(t) &= \int_0^t e^{\tau A} Bu(t-\tau)\, d\tau \\ &= \int_0^t \left(\sum_{i=0}^{n-1} \alpha_i(\tau) A^i \right) Bu(t-\tau)\, d\tau \\ &= \sum_{i=0}^{n-1} A^i B \int_0^t \alpha_i(\tau) u(t-\tau)\, d\tau = \mathcal{C}z \end{aligned}$$

where $z_i = \displaystyle\int_0^t \alpha_i(\tau) u(t-\tau)\, d\tau$. Hence, $x(t)$ is always in $\textbf{range}(\mathcal{C})$

Example

- unit masses at y_1, y_2, connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^{\mathsf{T}}\ \dot{y}^{\mathsf{T}}]^{\mathsf{T}}$



system is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u$$

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?

Example

controllability matrix is

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right] = \left[\begin{array}{cccc} 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right]$$

hence reachable set is

$$\mathcal{R} = \textbf{span}\left\{ \left[\begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ -1 \end{array}\right] \right\}$$

we can reach states with $y_1 = -y_2$, $\dot{y}_1 = -\dot{y}_2$, $i.e.$, precisely the differential motions

it's obvious — internal force does not affect center of mass position or total momentum!

Least-norm input for reachability

(also called *minimum energy input*)

assume that $\dot{x} = Ax + Bu$ is reachable

we seek u that steers $x(0) = 0$ to $x(t) = x_{\mathrm{des}}$ and minimizes

$$\int_0^t \|u(\tau)\|^2 \, d\tau$$

let's discretize system with interval $h = t/N$

(we'll let $N \to \infty$ later)

thus u is piecewise constant:

$$u(\tau) = u_d(k) \;\; \text{for} \;\; kh \leq \tau < (k+1)h, \qquad k = 0, \ldots, N-1$$

Least-norm input for reachability

so

$$x(t) = \left[\; B_d \;\; A_d B_d \;\; \cdots \;\; A_d^{N-1} B_d \;\right] \begin{bmatrix} u_d(N-1) \\ \vdots \\ u_d(0) \end{bmatrix}$$

where

$$A_d = e^{hA}, \quad B_d = \int_0^h e^{\tau A}\, d\tau B$$

least-norm u_d that yields $x(t) = x_{\mathrm{des}}$ is

$$u_{\mathrm{dln}}(k) = B_d^T (A_d^T)^{(N-1-k)} \left(\sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i \right)^{-1} x_{\mathrm{des}}$$

let's express in terms of A:

$$B_d^T (A_d^T)^{(N-1-k)} = B_d^T e^{(t-\tau)A^T}$$

where $\tau = t(k+1)/N$

Least-norm input for reachability

for N large, $B_d \approx (t/N)B$, so this is approximately

$$(t/N)B^\mathsf{T} e^{(t-\tau)A^\mathsf{T}}$$

similarly

$$\sum_{i=0}^{N-1} A_d^i B_d B_d^\mathsf{T} (A_d^\mathsf{T})^i = \sum_{i=0}^{N-1} e^{(ti/N)A} B_d B_d^\mathsf{T} e^{(ti/N)A^\mathsf{T}} \approx (t/N) \int_0^t e^{\bar{t}A} B B^\mathsf{T} e^{\bar{t}A^\mathsf{T}}\, d\bar{t}$$

for large N

hence least-norm discretized input is approximately

$$u_{\mathrm{ln}}(\tau) = B^\mathsf{T} e^{(t-\tau)A^\mathsf{T}} \left(\int_0^t e^{\bar{t}A} B B^\mathsf{T} e^{\bar{t}A^\mathsf{T}}\, d\bar{t} \right)^{-1} x_{\mathrm{des}}, \quad 0 \le \tau \le t$$

for large N

hence, this is the least-norm continuous input

- can make t small, but get larger u
- cf. DT solution: sum becomes integral

Minimum energy

min energy is

$$\int_0^t \|u_{\mathrm{ln}}(\tau)\|^2 \, d\tau = x_{\mathrm{des}}^\mathsf{T} Q(t)^{-1} x_{\mathrm{des}}$$

where

$$Q(t) = \int_0^t e^{\tau A} B B^\mathsf{T} e^{\tau A^\mathsf{T}} \, d\tau$$

can show

$$\begin{aligned}(A, B) \text{ controllable} &\Leftrightarrow & Q(t) > 0 \text{ for all } t > 0 \\ &\Leftrightarrow & Q(s) > 0 \text{ for some } s > 0\end{aligned}$$

in fact, **range**$(Q(t)) = \mathcal{R}$ for any $t > 0$

Minimum energy over infinite horizon

the matrix

$$P = \lim_{t\to\infty} \left(\int_0^t e^{\tau A} B B^\mathsf{T} e^{\tau A^\mathsf{T}}\, d\tau \right)^{-1}$$

always exists, and gives minimum energy required to reach a point x_{des} (with no limit on t):

$$\min \left\{ \int_0^t \|u(\tau)\|^2\, d\tau \;\middle|\; x(0) = 0,\ x(t) = x_{\mathrm{des}} \right\} = x_{\mathrm{des}}^\mathsf{T} P x_{\mathrm{des}}$$

- if A is stable, $P > 0$ ($i.e.$, can't get anywhere for free)
- if A is not stable, then P can have nonzero nullspace
- $Pz = 0$, $z \neq 0$ means can get to z using u's with energy as small as you like (u just gives a little kick to the state; the instability carries it out to z efficiently)

Reachability Gramian

if $\dot{x} = Ax + Bu$ is controllable and stable

then $W_r(t)$ converges as $t \to \infty$ to

$$W_r = \int_0^\infty e^{A\bar{t}} BB^\mathsf{T} e^{A^\mathsf{T}\bar{t}} \, d\bar{t},$$

the *reachability* (or *controllability*) *Gramian*. The CT reachability Gramian W_r satisfies the matrix equation

$$AW_r + W_r A^\mathsf{T} + BB^\mathsf{T} = 0$$

which is called the controllability *Lyapunov equation* to see this, note that

$$\frac{d}{dt} e^{tA} BB^\mathsf{T} e^{A^\mathsf{T}t} = Ae^{tA} BB^\mathsf{T} e^{A^\mathsf{T}t} + e^{tA} BB^\mathsf{T} e^{A^\mathsf{T}t} A^\mathsf{T}$$

integrate from $t = 0$ to ∞ to get:

$$e^{tA} BB^\mathsf{T} e^{A^\mathsf{T}t} \Big|_0^\infty = AW_r + W_r A^\mathsf{T}$$

which gives the Lyapunov equation (a linear equation in W_r which can be efficiently solved)

General state transfer

consider state transfer from $x(t_i)$ to $x(t_f) = x_{\text{des}}$, $t_f > t_i$

since

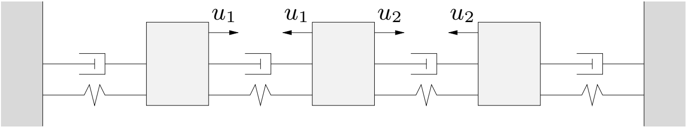
$$x(t_f) = e^{(t_f - t_i)A}x(t_i) + \int_{t_i}^{t_f} e^{(t_f - \tau)A}Bu(\tau)\, d\tau$$

u steers $x(t_i)$ to $x(t_f) = x_{\text{des}}$ $\Leftrightarrow$

u (shifted by t_i) steers $x(0) = 0$ to $x(t_f - t_i) = x_{\text{des}} - e^{(t_f - t_i)A}x(t_i)$

- general state transfer reduces to reachability problem
- if system is controllable, any state transfer can be effected
  - in 'zero' time with impulsive inputs
  - in any positive time with non-impulsive inputs

Example



- unit masses, springs, dampers
- u_1 is force between 1st & 2nd masses
- u_2 is force between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3, state $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

system is:

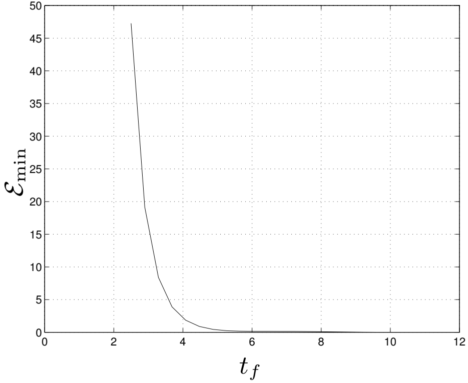
$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example

steer state from $x(0) = e_1$ to $x(t_f) = 0$

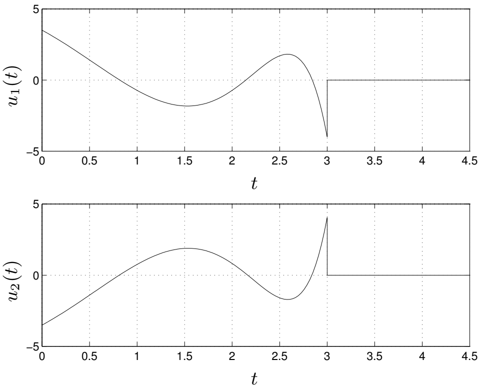
i.e., control initial state e_1 to zero at $t = t_f$

$\mathcal{E}_{\mathrm{min}} = \int_0^{t_f} \|u_{\mathrm{ln}}(\tau)\|^2 \, d\tau$ vs. t_f:



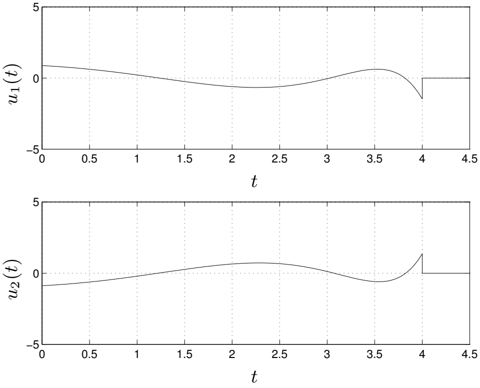
Example

for $t_f = 3$, $u = u_{\mathtt{ln}}$ is:



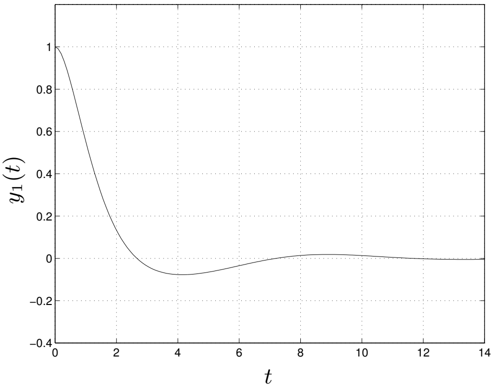
Example

and for $t_f = 4$:



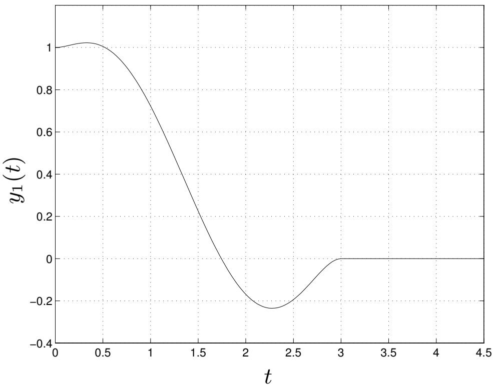
Example

output y_1 for $u = 0$:



Example

output y_1 for $u = u_{\mathtt{ln}}$ with $t_f = 3$:



Example

output y_1 for $u = u_{\mathtt{ln}}$ with $t_f = 4$: