15.2270. Eigenvalues and singular values of a symmetric matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues, and let $\sigma_1, \ldots, \sigma_n$ be the singular values of a matrix $A \in \mathbb{R}^{n \times n}$, which satisfies $A = A^T$. (The singular values are based on the full SVD: If rank($A$) < $n$, then some of the singular values are zero.) You can assume the eigenvalues (and of course singular values) are sorted, i.e., $\lambda_1 \geq \cdots \geq \lambda_n$ and $\sigma_1 \geq \cdots \geq \sigma_n$. How are the eigenvalues and singular values related?

15.2280. More facts about singular values of matrices. For each of the following statements, prove it if it is true; otherwise give a specific counterexample. Here $X, Y, Z \in \mathbb{R}^{n \times n}$.

a) $\sigma_{\text{max}}(X) \geq \max_{1 \leq i \leq n} \sqrt{\sum_{1 \leq j \leq n} |X_{ij}|^2}$.

b) $\sigma_{\text{min}}(X) \geq \min_{1 \leq i \leq n} \sqrt{\sum_{1 \leq j \leq n} |X_{ij}|^2}$.

c) $\sigma_{\text{max}}(XY) \leq \sigma_{\text{max}}(X)\sigma_{\text{max}}(Y)$.

d) $\sigma_{\text{min}}(XY) \geq \sigma_{\text{min}}(X)\sigma_{\text{min}}(Y)$.

e) $\sigma_{\text{min}}(X+Y) \geq \sigma_{\text{min}}(X) - \sigma_{\text{max}}(Y)$.

15.2790. Ellipsoids.

a) Write a function that, given a a $2 \times 2$ real symmetric matrix $A$, plots the ellipse

$$E = \{ x \in \mathbb{R}^2 \mid x^T Ax = 1 \}$$

Make sure that your plot is shown so that horizontal and vertical lengths are the same, that is, with aspect ratio 1. Turn in your code.

**Julia hint:** use the following to draw a plot with correct aspect ratio.

Using Plots; plot(x, y, aspect_ratio=:equal)

b) Use your code to plot the ellipsoid for the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

c) Use your code to plot the ellipsoid for the matrix

$$A = \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.4 \end{bmatrix}$$

On your plot, also show semiaxes.

d) Consider an estimation problem, where we have three sensors, define by $b_i \in \mathbb{R}^2$ for $i = 1, 2, 3$. We measure $y_i = b_i^T x$. The vectors $b_i$ are

$$b_1 = \begin{bmatrix} 0.89 \\ 0.45 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0.45 \\ 0.89 \end{bmatrix} \quad b_3 = \begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}$$

Plot the set of $x \in \mathbb{R}^2$ for which $\|y\| \leq 1$. On your plot, show also the $b_i$ (that is, plot a line from the origin to $b_i$).
16.2670. **Regularization and SVD.** Let $A \in \mathbb{R}^{n \times n}$ be full rank, with SVD

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T.$$  

(We consider the square, full rank case just for simplicity; it’s not too hard to consider the general nonsquare, non-full rank case.) Recall that the regularized approximate solution of $Ax = y$ is defined as the vector $x_{\text{reg}} \in \mathbb{R}^n$ that minimizes the function

$$\|Ax - y\|^2 + \mu \|x\|^2,$$

where $\mu > 0$ is the regularization parameter. The regularized solution is a linear function of $y$, so it can be expressed as $x_{\text{reg}} = By$ where $B \in \mathbb{R}^{n \times n}$.

a) Express the SVD of $B$ in terms of the SVD of $A$. To be more specific, let

$$B = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^T$$

denote the SVD of $B$. Express $\tilde{\sigma}_i, \tilde{u}_i, \tilde{v}_i$, for $i = 1, \ldots, n$, in terms of $\sigma_i, u_i, v_i, i = 1, \ldots, n$ (and, possibly, $\mu$). Recall the convention that $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_n$.

b) Find the norm of $B$. Give your answer in terms of the SVD of $A$ (and $\mu$).

c) Find the worst-case relative inversion error, defined as

$$\max_{y \neq 0} \frac{\|ABy - y\|}{\|y\|}.$$  

Give your answer in terms of the SVD of $A$ (and $\mu$).

16.2980. **Smoothing.** We have a discrete-time signal given by $x \in \mathbb{R}^n$. We get to measure $y \in \mathbb{R}^n$, given by

$$y_i = \sum_{k=-h}^{h} c_k x_{i+k} + w_i \quad \text{for } i = 1, \ldots, n$$

where $w_i$ is noise. Here we use the convention that $x_i = 0$ for $i < 1$ or $i > n$. That is, $y$ is $c$ convolved with $x$ plus noise. In applications, very often the effect of convolution with $c$ is to smooth or blur $x$, and we would like to undo this.

The file *regl_data.json* contains $c$, $w$ and $x$.

a) In Julia, construct the $n \times n$ matrix such that $y = Ax + w$. Plot the singular values $\sigma_k$ against $k$.

b) Plot the first 6 right singular vectors of $A$ (i.e. plot $V_{ij}$ against $i$ for $j = 1, \ldots, 6$.) Explain what you see.

c) Find and plot the least-squares estimate of $x$ given $y_{\text{meas}}$. Explain what happens.
Many of the singular values of $A$ are very small; this means that the measurement in the directions of the corresponding right singular vectors is being swamped by the noise.

If we believe these components are small, we can remove them from our estimate of $x$ altogether by truncating the SVD of $A$ and using the truncated SVD to compute the estimate. This is called the truncated SVD regularization of least-squares.

Suppose we decided only to keep the first $r$ components. Then truncate by letting $\tilde{V}$ and $\tilde{U}$ be the first $r$ columns of $V$ and $U$, and letting $\tilde{\Sigma}$ be the top-left $r \times r$ submatrix of $\Sigma$. Then we can construct an estimator that ignores the noise components by

$$A_{\text{est}} = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T$$

and set

$$x_{\text{est}} = A_{\text{est}}y_{\text{meas}}$$

For values of $r$ in $5, 10, 15, 30, 50$, compute and plot the corresponding estimates of $x$. Explain what you see.

e) For each $r$ between 1 and 35, compute the norm of the error $\|x - x_{\text{est}}\|$.

Plot this against $r$. Explain what you see.

f) Pick the ‘best’ $r$ and plot the corresponding estimate.

g) Another approach is to use Tychonov regularization. Find and plot the vector $x_{\text{reg}} \in \mathbb{R}^n$ that minimizes the function

$$\|Ax - y\|^2 + \mu\|x\|^2,$$

where $\mu > 0$ is the regularization parameter. Pick a value of $\mu$ that gives a good estimate, in your opinion.

h) The regularized solution is a linear function of $y$, so it can be expressed as $x_{\text{reg}} = By$ where $B \in \mathbb{R}^{n \times n}$. Express the SVD of $B$ in terms of the SVD of $A$. To be more specific, let

$$B = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^T$$

denote the SVD of $B$. Express $\tilde{\sigma}_i, \tilde{u}_i, \tilde{v}_i, i = 1, \ldots, n$, in terms of $\sigma_i, u_i, v_i, i = 1, \ldots, n$ (and, possibly, $\mu$). Recall the convention that $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_n$.

i) Find the norm of $B$. Give your answer in terms of the SVD of $A$ (and $\mu$).

j) Find the worst-case relative inversion error, defined as

$$\max_{y \neq 0} \frac{\|ABy - y\|}{\|y\|}.$$