EE263 Homework 7

Fall 2023

8.1320. Portfolio selection with sector neutrality constraints. We consider the problem of selecting a portfolio composed of n assets. We let $x_i \in \mathbb{R}$ denote the investment (say, in dollars) in asset i, with $x_i < 0$ meaning that we hold a short position in asset i. We normalize our total portfolio as $\mathbf{1}^\mathsf{T} x = 1$, where $\mathbf{1}$ is the vector with all entries 1. (With normalization, the x_i are sometimes called *portfolio weights*.)

The portfolio (mean) return is given by $r = \mu^\mathsf{T} x$, where $\mu \in \mathbb{R}^n$ is a vector of asset (mean) returns. We want to choose x so that r is large, while avoiding risk exposure, which we explain next.

First we explain the idea of *sector exposure*. We have a list of k economic sectors (such as manufacturing, energy, transportation, defense, ...). A matrix $F \in \mathbb{R}^{k \times n}$, called the *factor loading matrix*, relates the portfolio x to the *factor exposures*, given as $R^{\text{fact}} = Fx \in \mathbb{R}^k$. The number R_i^{fact} is the portfolio risk exposure to the ith economic sector. If R_i^{fact} is large (in magnitude) our portfolio is exposed to risk from changes in that sector; if it is small, we are less exposed to risk from that sector. If $R_i^{\text{fact}} = 0$, we say that the portfolio is *neutral* with respect to sector i.

Another type of risk exposure is due to fluctations in the returns of the individual assets. The *idiosyncratic risk* is given by

$$R^{\text{id}} = \sum_{i=1}^{n} \sigma_i^2 x_i^2,$$

where $\sigma_i > 0$ are the standard deviations of the asset returns. (You can take the formula above as a definition; you do not need to understand the statistical interpretation.)

We will choose the portfolio weights x so as to maximize $r - \lambda R^{\text{id}}$, which is called the *risk-adjusted return*, subject to neutrality with respect to all sectors, *i.e.*, $R^{\text{fact}} = 0$. Of course we also have the normalization constraint $\mathbf{1}^\mathsf{T} x = 1$. The parameter λ, which is positive, is called the *risk aversion parameter*. The (known) data in this problem are $\mu \in \mathbb{R}^n$, $F \in \mathbb{R}^{k \times n}$, $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$.

a) Explain how to find x, using methods from the course. You are welcome (even encouraged) to express your solution in terms of block matrices, formed from the given data.

b) Using the data given in `sector_neutral_portfolio_data.json`, find the optimal portfolio. Report the associated values of r (the return), and R^{id} (the idiosyncratic risk). Verify that $\mathbf{1}^\mathsf{T} x = 1$ (or very close) and $R^{\text{fact}} = 0$ (or very small).

9.1360. A simple population model. We consider a certain population of fish (say) each (yearly) season. $x(t) \in \mathbb{R}^3$ will describe the population of fish at year $t \in \mathbb{Z}$, as follows:

- $x_1(t)$ denotes the number of fish less than one year old
- $x_2(t)$ denotes the number of fish between one and two years old
- $x_3(t)$ denotes the number of fish between two and three years

(We will ignore the fact that these numbers are integers.) The population evolves from year t to year $t+1$ as follows.

- The number of fish less than one year old in the next year $(t+1)$ is equal to the total number of offspring born during the current year. Fish that are less than one year old in the current year (t) bear no offspring. Fish that are between one and two years old in the current year (t) bear an average of 2 offspring each. Fish that are between two and three years old in the current year (t) bear an average of 1 offspring each.
- 40% of the fish less than one year old in the current year (t) die; the remaining 60% live on to be between one and two years old in the next year $(t+1)$.
- 30% of the one-to-two year old fish in the current year die, and 70% live on to be two-to-three year old fish in the next year.
- All of the two-to-three year old fish in the current year die.

Express the population dynamics as an autonomous linear system with state $x(t)$, *i.e.*, in the form $x(t+1) = Ax(t)$. **Remark:** this example is silly, but more sophisticated population dynamics models are very useful and widely used.

9.1410. Invariance of the unit square. Consider the linear dynamical system $\dot{x} = Ax$ with $A \in \mathbb{R}^{2\times 2}$. The unit square in $\mathbb{R}^2$ is defined by

$$S = \{\ x \mid -1 \le x_1 \le 1,\ \ -1 \le x_2 \le 1\ \}.$$

a) Find the exact conditions on A for which the unit square S is invariant under $\dot{x} = Ax$. Give the conditions as explicitly as possible.

b) Consider the following statement: if the eigenvalues of A are real and negative, then S is invariant under $\dot{x} = Ax$. Either show that this is true, or give an explicit counterexample.

9.1470. Optimal choice of initial temperature profile. We consider a thermal system described by an n-element finite-element model. The elements are arranged in a line, with the temperature of element i at time t denoted $T_i(t)$. Temperature is measured in degrees Celsius above ambient; negative $T_i(t)$ corresponds to a temperature below ambient. The dynamics of the system are described by

$$c_1\dot{T}_1 = -a_1T_1 - b_1(T_1 - T_2),$$

$$c_i\dot{T}_i = -a_iT_i - b_i(T_i - T_{i+1}) - b_{i-1}(T_i - T_{i-1}), \quad i = 2, \ldots, n-1,$$

and

$$c_n\dot{T}_n = -a_nT_n - b_{n-1}(T_n - T_{n-1}).$$

where $c \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, and $b \in \mathbb{R}^{n-1}$ are given and are all positive.

We can interpret this model as follows. The parameter c_i is the heat capacity of element i, so $c_i\dot{T}_i$ is the net heat flow into element i. The parameter a_i gives the thermal conductance between element i and the environment, so a_iT_i is the heat flow from element i to the environment (*i.e.*, the direct heat loss from element i.) The parameter b_i gives the thermal

conductance between element i and element $i+1$, so $b_i(T_i - T_{i+1})$ is the heat flow from element i to element $i+1$. Finally, $b_{i-1}(T_i - T_{i-1})$ is the heat flow from element i to element $i-1$.

The goal of this problem is to choose the initial temperature profile, $T(0) \in \mathbb{R}^n$, so that $T(t^{\text{des}}) \approx T^{\text{des}}$. Here, $t^{\text{des}} \in \mathbb{R}$ is a specific time when we want the temperature profile to closely match $T^{\text{des}} \in \mathbb{R}^n$. We also wish to satisfy a constraint that $\|T(0)\|$ should be not be too large.

To formalize these requirements, we use the objective $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$ and the constraint $(1/\sqrt{n})\|T(0)\| \leq T^{\max}$. The first expression is the RMS temperature deviation, at $t = t^{\text{des}}$, from the desired value, and the second is the RMS temperature deviation from ambient at $t = 0$. $T^{\max}$ is the (given) maximum inital RMS temperature value.

a) Explain how to find $T(0)$ that minimizes the objective while satisfying the constraint.

b) Solve the problem instance with the values of n, c, a, b, t_{des}, T^{des} and $T^{\max}$ defined in the file `temp_prof_data.json`.

   Plot, on one graph, your $T(0)$, $T(t^{\text{des}})$ and T^{des}. Give the RMS temperature error $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$, and the RMS value of initial temperature $(1/\sqrt{n})\|T(0)\|$.

10.1500. Properties of the matrix exponential.

a) Show that $e^{A+B} = e^A e^B$ if A and B commute, *i.e.*, $AB = BA$.

b) Carefully show that $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$.

11.1790. Squareroot and logarithm of a (diagonalizable) matrix. Suppose that $A \in \mathbb{R}^{n\times n}$ is diagonalizable. Therefore, an invertible matrix $T \in \mathbb{C}^{n\times n}$ and diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ exist such that $A = T\Lambda T^{-1}$. Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

a) We say $B \in \mathbb{R}^{n\times n}$ is a squareroot of A if $B^2 = A$. Let μ_i satisfy $\mu_i^2 = \lambda_i$. Show that $B = T \operatorname{diag}(\mu_1, \ldots, \mu_n)T^{-1}$ is a squareroot of A. A squareroot is sometimes denoted $A^{1/2}$ (but note that there are in general many squareroots of a matrix). When λ_i are real and nonnegative, it is conventional to take $\mu_i = \sqrt{\lambda_i}$ (*i.e.*, the nonnegative squareroot), so in this case $A^{1/2}$ is unambiguous.

b) We say B is a logarithm of A if $e^B = A$, and we write $B = \log A$. Following the idea of part a, find an expression for a logarithm of A (which you can assume is invertible). Is the logarithm unique? What if we insist on B being real?

15.2380. Recovering an ellipsoid from boundary points. You are given a set of vectors $x^{(1)}, \ldots, x^{(N)} \in \mathbb{R}^n$ that are thought to lie on or near the surface of an ellipsoid centered at the origin, which we represent as

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid x^\mathsf{T} A x = 1\},$$

where $A = A^\mathsf{T} \in \mathbb{R}^{n\times n} \geq 0$. Your job is to recover, at least approximately, the matrix A, given the observed data $x^{(1)}, \ldots, x^{(N)}$. Explain your approach to this problem, and then carry it out on the data given in the mfile `ellip_bdry_data.json`. Be sure to explain how you check that the ellipsoid you find is reasonably consistent with the given data, and also that the matrix A

you find does, in fact, correspond to an ellipsoid. To simplify the explanation, you can give it for the case $n = 4$ (which is the dimension of the given data). But it should be clear from your discussion how it works in general.