1. Harmonic oscillator. The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a harmonic oscillator.

a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

b) Sketch the vector field of the harmonic oscillator.

c) The state trajectories describe circular orbits, i.e., $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

Solution.

a) We have

$$(sI - A)^{-1} = \frac{1}{s^2 + \omega^2} \begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}.$$

From this result it follows that the eigenvalues of $A$ are given by $\{\pm j\omega\}$. The inverse Laplace transform gives

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$$

and we have $x(t) = \Phi(t)x(0)$.

b) Here is the vector field:

c) First we note from basic trigonometric relations that $\Phi^T(t)\Phi(t) = I$. From this we conclude that $\Phi(t)$ is orthogonal. Now it follows that $x^T(t)x(t) = x^T(0)\Phi^T(t)\Phi(t)x(0) = x^T(0)x(0)$, i.e. $\|x(t)\| = \|x(0)\|$.

d) Using previous relations we can write

$$\dot{x}^T x = x^T \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x = \begin{bmatrix} -\omega x_2 & \omega x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This shows that the velocity vector is always orthogonal to the position vector, as claimed.
2. Real modal form. We learned about the modal form of a system in class. Show that when some of eigenvalues of the dynamics matrix $A$ are complex, the system can be put in real modal form (Assuming the eigenvectors of $A$ are independent):

$$S^{-1}AS = \text{diag} (\Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1})$$

where $\Lambda_r = \text{diag}(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r + 1, r + 3, \ldots, n - 1$$

where $\lambda_j$ are the complex eigenvalues (one from each conjugate pair). Clearly explain what the matrix $S$ is.

Generate a matrix $A$ in $\mathbb{R}^{10 \times 10}$ using $A=\text{randn}(10)$. (The entries of $A$ will be drawn from a unit normal distribution.) Find the eigenvalues of $A$. If by chance they are all real, generate a new instance of $A$. Find the real modal form of $A$, i.e., a matrix $S$ such that $S^{-1}AS$ has the real modal form. Your solution should include the source code that you use to find $S$, and some code that checks the results (i.e., computes $S^{-1}AS$ to verify it has the required form).

Solution. Assuming $A$ is diagonalizable, it can be written as $A = T\Lambda T^{-1}$, Here

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n),$$

where $\lambda_1, \ldots, \lambda_r$ are the real eigenvalues of $A$ and $\lambda_{r+1}, \ldots, \lambda_n$ are the complex eigenvalues of $A$ and come in complex conjugate pairs. Let $t_i$ be the $i$th column of $T$. Take $S$ to be

$$S = [t_1 \cdots t_r \ \Re(t_{r+1}) \ \Im(t_{r+1}) \ \cdots \ \Re(t_{n-1}) \ \Im(t_{n-1})].$$
Let us now prove why constructing $S$ in this way will give us the desired result. Let $v = \Re(v) + i\Im(v)$ be a complex eigenvector of $A$ associated with the eigenvalue $\lambda = \sigma + i\omega$. We must have $Av = \lambda v$, i.e.,

$$A(\Re(v) + i\Im(v)) = (\sigma + i\omega)(\Re(v) + i\Im(v)).$$

This implies that

$$A\Re(v) = \sigma \Re(v) - \omega \Im(v), \quad A\Im(v) = \omega \Re(v) + \sigma \Im(v),$$

or equivalently

$$A[\Re(v) \quad \Im(v)] = [\Re(v) \quad \Im(v)] \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

If we do the same derivation for the other complex conjugate eigenvalue pairs, we get the construction of $S$ that was presented above.

Here is a short MATLAB script that checks that our solution is correct.

```matlab
randn('seed', 21395);
A = randn(10);
[V, D] = eig(A);
% Take eigenvectors of complex eigenvalues and arrange them in pairs.
S = zeros(10);
S(:,1) = V(:,5);
S(:,2) = V(:,6);
S(:,3) = V(:,7);
S(:,4) = V(:,10);
S(:,5) = real(V(:,1));
S(:,6) = imag(V(:,1));
S(:,7) = real(V(:,3));
S(:,8) = imag(V(:,3));
S(:,9) = real(V(:,8));
S(:,10) = imag(V(:,8));

% Inspect S^{-1}AS
inv(S)*A*S
```


a) Show that $e^{A+B} = e^A e^B$ if $A$ and $B$ commute, i.e., $AB = BA$.

b) Carefully show that $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$. 

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Solution.

a) We will show that if $A$ and $B$ commute then $e^A e^B = e^{A+B}$. We begin by writing the expressions for $e^A$ and $e^B$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

Now we multiply both expressions and get

$$e^A e^B = I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} + \frac{A^3}{3!} + \frac{A^2 B}{2!} + \frac{AB^2}{2!} + \frac{B^3}{3!} + \cdots$$

Now we note that, if $A$ and $B$ commute, we are able to write things such as $(A + B)^2 = A^2 + 2AB + B^2$. So, if $A$ and $B$ commute we can finally write

$$e^A e^B = I + (A + B) + \frac{(A + B)^2}{2!} + \frac{(A + B)^3}{3!} + \cdots = e^{A+B}$$

b) It suffices to note that $A$ commute with itself. Then one can write

$$\frac{d e^{At}}{dt} = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots$$

$$= A (I + At + \frac{(At)^2}{2!} + \cdots)$$

$$= (I + At + \frac{(At)^2}{2!} + \cdots)A$$

$$= Ae^{At} = e^{At} A$$

4. Output response envelope for linear system with uncertain initial condition. We consider the autonomous linear dynamical system $\dot{x} = Ax$, $y(t) = Cx(t)$, where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$. We do not know the initial condition exactly; we only know that it lies in a ball of radius $r$ centered at the point $x_0$:

$$\|x(0) - x_0\| \leq r.$$  

We call $x_0$ the nominal initial condition, and the resulting output, $y_{nom}(t) = Ce^{tA}x_0$, the nominal output. We define the maximum output or upper output envelope as

$$\overline{y}(t) = \max\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the maximum possible value of the output at time $t$, over all possible initial conditions. (Here you can choose a different initial condition for each $t$; you are not required to find a single initial condition.) In a similar way, we define the minimum output or lower output envelope as

$$\underline{y}(t) = \min\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the minimum possible value of the output at time $t$, over all possible initial conditions.

a) Explain how to find $\overline{y}(t)$ and $\underline{y}(t)$, given the problem data $A$, $C$, $x_0$, and $r$. 

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b) Carry out your method on the problem data in uie_data.m. On the same axes, plot $y_{\text{nom}}$, $\bar{y}$, and $\underline{y}$ versus $t$, over the range $0 \leq t \leq 10$.

**Solution.** We have

$$y(t) = Cx(t) = Ce^{tA}x(0) = Ce^{tA}(x(0) - x_0) = y_{\text{nom}}(t) + Ce^{tA}(x(0) - x_0).$$

Using the Cauchy-Schwarz inequality, we get

$$-\|\left(e^{tA}\right)^{T}C^{T}\|\|x(0) - x_0\| \leq Ce^{tA}(x(0) - x_0) \leq \|\left(e^{tA}\right)^{T}C^{T}\|\|x(0) - x_0\|.$$

However, since $\|x(0) - x_0\| \leq r$, we can deduce that

$$y_{\text{nom}}(t) - \|\left(e^{tA}\right)^{T}C^{T}\|r \leq y(t) \leq y_{\text{nom}}(t) + \|\left(e^{tA}\right)^{T}C^{T}\|r.$$

In fact, these inequalities are tight. This is so, since if

$$x(0) = x_0 + \frac{r}{\|\left(e^{tA}\right)^{T}C^{T}\|}\left(e^{tA}\right)^{T}C^{T}$$

then

$$y(t) = y_{\text{nom}}(t) + \|\left(e^{tA}\right)^{T}C^{T}\|r.$$

Similarly, if

$$x(0) = x_0 - \frac{r}{\|\left(e^{tA}\right)^{T}C^{T}\|}\left(e^{tA}\right)^{T}C^{T}$$

then

$$y(t) = y_{\text{nom}}(t) - \|\left(e^{tA}\right)^{T}C^{T}\|r.$$

Therefore we finally have

$$\bar{y}(t) = y_{\text{nom}}(t) - \|\left(e^{tA}\right)^{T}C^{T}\|r, \quad \underline{y}(t) = y_{\text{nom}}(t) + \|\left(e^{tA}\right)^{T}C^{T}\|r.$$

The following matlab code performs this for the given data:

```matlab
clear all
uie_data
t = linspace(0,10,1000);
y_nom = [];
y_over = [];
y_under = [];
for k = t
    v = C*expm(k*A);
y_c = v*x_0;
y_nom = [y_nom y_c];
y_under = [y_under y_c-norm(v)*r];
y_over = [y_over y_c+norm(v)*r];
end
figure
plot(t,y_nom,'b-',t,y_under,'r--',t,y_over,'r--')
```

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This code generates the following figure:

Note that the bounds actually move towards each other near $t \approx 5$, but then diverge again.

5. **Spectral mapping theorem.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is analytic, i.e., given by a power series expansion

$$f(u) = a_0 + a_1u + a_2u^2 + \cdots$$

(where $a_i = f^{(i)}(0)/(i!)$). (You can assume that we only consider values of $u$ for which this series converges.) For $A \in \mathbb{R}^{n \times n}$, we define $f(A)$ as

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots$$

(again, we’ll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbb{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if $\lambda$ is an eigenvalue of $A$, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the *spectral mapping theorem*.

To illustrate this with an example, generate a random $3 \times 3$ matrix, for example using $A = \text{randn}(3)$. Find the eigenvalues of $(I + A)(I - A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various computations.)

**Solution.** Assuming $Av = \lambda v$, $v \neq 0$, we have

$$f(A)v = a_0Iv + a_1Av + a_2A^2v + \cdots$$

$$= a_0v + a_1\lambda v + a_2\lambda^2v + \cdots$$

$$= f(\lambda)v,$$
using $A^kv = \lambda^kv$.

The matlab code that illustrates this is shown below.

```matlab
randn('state',0); % makes it repeatable
A=randn(3);
% eigenvalues of A
lambdas=eig(A);
% create matrix f(A)=(I+A)(I-A)^(-1)
fA=(eye(3)+A)*inv(eye(3)-A);
% eigenvalues of f(A)
lambdas_fA=eig(fA);
% eigenvalues of B via spectral mapping theorem
lambdas_fA_smt = (1+lambdas)./(1-lambdas);
% compare (need not be in same order!)
[lambdas_fA lambdas_fA_smt]
```

6. **Interconnection of linear systems.** Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are themselves linear systems. In this problem you consider the specific interconnection shown below:

Here, there are two subsystems $S$ and $T$. Subsystem $S$ is characterized by

$$\dot{x} = Ax + B_1u + B_2w_1,$$

$$w_2 =Cx + D_1u + D_2w_1,$$

and subsystem $T$ is characterized by

$$\dot{z} = Fz + G_1v + G_2w_2,$$

$$w_1 = H_1z,$$

$$y = H_2z + Jw_2.$$

We don’t specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (i.e., compatible) dimensions. Note that the subscripts in the matrices above, as in $B_1$ and $B_2$, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \quad \begin{bmatrix} x \\ z \end{bmatrix}, \quad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making.
Solution. This one is easier than it might appear. All we need to do is write down all the equations for this system, and massage them to be in the form of a linear dynamical system with the given input, state, and output. Substituting the expression for \( w_1 \) into the first set of equations gives

\[
\dot{x} = Ax + B_1u + B_2H_1z, \quad w_2 = Cx + D_1u + D_2H_1z.
\]

Similarly, substituting the expression for \( w_2 \) into the second set of equations yields

\[
\dot{z} = Fz + G_1v + G_2(Cx + D_1u + D_2H_1z), \quad y = H_2z + J(Cx + D_1u + D_2H_1z).
\]

Now we just put this together into the required form:

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
A & B_2H_1 \\
G_2C & F + G_2D_2H_1
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
B_1 & 0 \\
G_2D_1 & G_1
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\]

and

\[
y =
\begin{bmatrix}
JC & H_2 + JD_2H_1
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
JD_1 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\]

7. Analysis of investment allocation strategies. Each year or period (denoted \( t = 0, 1, \ldots \)) an investor buys certain amounts of one-, two-, and three-year certificates of deposit (CDs) with interest rates 5%, 6%, and 7%, respectively. (We ignore minimum purchase requirements, and assume they can be bought in any amount.)

- \( B_1(t) \) denotes the amount of one-year CDs bought at period \( t \).
- \( B_2(t) \) denotes the amount of two-year CDs bought at period \( t \).
- \( B_3(t) \) denotes the amount of three-year CDs bought at period \( t \).

We assume that \( B_1(0) + B_2(0) + B_3(0) = 1 \), i.e., a total of 1 is to be invested at \( t = 0 \). (You can take \( B_j(t) \) to be zero for \( t < 0 \).) The total payout to the investor, \( p(t) \), at period \( t \) is a sum of six terms:

- \( 1.05B_1(t-1) \), i.e., principle plus 5% interest on the amount of one-year CDs bought one year ago.
- \( 1.06B_2(t-2) \), i.e., principle plus 6% interest on the amount of two-year CDs bought two years ago.
- \( 1.07B_3(t-3) \), i.e., principle plus 7% interest on the amount of three-year CDs bought three years ago.
- \( 0.06B_2(t-1) \), i.e., 6% interest on the amount of two-year CDs bought one year ago.
- \( 0.07B_3(t-1) \), i.e., 7% interest on the amount of three-year CDs bought one year ago.
- \( 0.07B_3(t-2) \), i.e., 7% interest on the amount of three-year CDs bought two years ago.
The total wealth held by the investor at period \( t \) is given by

\[
w(t) = B_1(t) + B_2(t) + B_2(t-1) + B_3(t) + B_3(t-1) + B_3(t-2).
\]

Two re-investment allocation strategies are suggested.

- **The 35-35-30 strategy.** The total payout is re-invested 35\% in one-year CDs, 35\% in two-year CDs, and 30\% in three-year CDs. The initial investment allocation is the same: \( B_1(0) = 0.35, B_2(0) = 0.35, \) and \( B_3(0) = 0.30 \).

- **The 60-20-20 strategy.** The total payout is re-invested 60\% in one-year CDs, 20\% in two-year CDs, and 20\% in three-year CDs. The initial investment allocation is:

\[
B_1(0) = 0.60, B_2(0) = 0.20, \text{ and } B_3(0) = 0.20.
\]

a) Describe the investments over time as a linear dynamical system

\[
x(t+1) = Ax(t),
\]

\[
y(t) = Cx(t)
\]

with \( y(t) \) equal to the total wealth at time \( t \). Be very clear about what the state \( x(t) \) is, and what the matrices \( A \) and \( C \) are. You will have two such linear systems: one for the 35-35-30 strategy and one for the 60-20-20 strategy.

b) **Asymptotic wealth growth rate.** For each of the two strategies described above, determine the asymptotic growth rate, defined as \( \lim_{t \to \infty} w(t+1)/w(t) \). (If this limit doesn’t exist, say so.) *Note:* simple numerical simulation of the strategies (e.g., plotting \( w(t+1)/w(t) \) versus \( t \) to guess its limit) is not acceptable. (You can, of course, simulate the strategies to check your answer.)

c) **Asymptotic liquidity.** The total wealth at time \( t \) can be divided into three parts:

- \( B_1(t) + B_2(t-1) + B_3(t-2) \) is the amount that matures in one year (i.e., the amount of principle we could get back next year)
- \( B_2(t) + B_3(t-1) \) is the amount that matures in two years
- \( B_3(t) \) is the amount that matures in three years (i.e., is least liquid)

We define liquidity ratios as the ratio of these amounts to the total wealth:

\[
L_1(t) = (B_1(t) + B_2(t-1) + B_3(t-2))/w(t),
\]

\[
L_2(t) = (B_2(t) + B_3(t-1))/w(t),
\]

\[
L_3(t) = B_3(t)/w(t).
\]

For the two strategies above, do the liquidity ratios converge as \( t \to \infty \)? If so, to what values? *Note:* as above, simple numerical simulation alone is not acceptable.

d) Suppose you could change the initial investment allocation for the 35-35-30 strategy, i.e., choose some other nonnegative values for \( B_1(0), B_2(0), \) and \( B_3(0) \) that satisfy \( B_1(0) + B_2(0) + B_3(0) = 1 \). What allocation would you pick, and how would it be better than the (0.35, 0.35, 0.30) initial allocation? (For example, would the asymptotic growth rate be larger?) How much better is your choice of initial investment allocations? *Hint for part d:* think very carefully about this one. *Hint for whole problem:* watch out for nondiagonalizable, or nearly nondiagonalizable, matrices. Don’t just blindly type in matlab commands; check to make sure you’re computing what you think you’re computing.
Solution.

a) We take as state vector
\[ x(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_2(t - 1) \\ B_3(t) \\ B_3(t - 1) \\ B_3(t - 2) \end{bmatrix}. \]

The components consist of the six critical quantities: the amount of one-, two-, and three-year CDs held of each possible maturity date (i.e., one, two, and three years). We can express the system as
\[ x(t + 1) = Fx(t) + Gu(t) \]
where \( u(t) \in \mathbb{R}^3 \) gives the amount of one, two, and three year CDs purchased at \( t + 1 \), and
\[
F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The payout is given by \( p(t) = hx(t) \), where
\[ h = [1.05 \ 0.06 \ 1.06 \ 0.07 \ 0.07 \ 1.07]. \]

The allocation of the payout is given by \( u(t) = [0.35 \ 0.35 \ 0.30]^T p(t) \) for the 35-35-30 strategy and by \( u(t) = [0.60 \ 0.20 \ 0.20]^T p(t) \) for the 60-20-20 strategy. Finally, putting it all together, we end up with \( x(t + 1) = Ax(t) \) where
\[
A_{35} = F + G[0.35 \ 0.35 \ 0.30]^T h
\]
for the 35-35-30 case, and similarly for the 60-20-20 case. To be fully explicit, we have
\[
A_{35} = \begin{bmatrix} 0.3675 & 0.0210 & 0.3710 & 0.0245 & 0.0245 & 0.3745 \\ 0.3675 & 0.0210 & 0.3710 & 0.0245 & 0.0245 & 0.3745 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0.3150 & 0.0180 & 0.3180 & 0.0210 & 0.0210 & 0.3210 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix},
\]
\[
A_{60} = \begin{bmatrix} 0.6300 & 0.0360 & 0.6360 & 0.0420 & 0.0420 & 0.6420 \\ 0.2100 & 0.0120 & 0.2120 & 0.0140 & 0.0140 & 0.2140 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0.2100 & 0.0120 & 0.2120 & 0.0140 & 0.0140 & 0.2140 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix}.
\]

The following matlab code calculates the matrix \( A \) for each allocation strategy.

\[
AA = [0 \ 0 \ 0 \ 0 \ 0 \ 0];
\]
BB = [1 0 0; 0 1 0; 0 0 1];
PP = [1.05 0.06 1.06 0.07 0.07 1.07]; % PP*x gives payout
alloc35 = [.35 .35 .3]'; % for 35-35-30 allocation
A35 = AA+BB*alloc35*PP;
alloc60 = [0.60 0.20 0.20]'; % for 60-20-20 allocation
A60 = AA+BB*alloc60*PP;
WW=ones(1,6); %WW*x gives total wealth

This yields:

A35 =
0.3675 0.0210 0.3710 0.0245 0.0245 0.3745
0.3675 0.0210 0.3710 0.0245 0.0245 0.3745
0 1.0000 0 0 0 0
0.3150 0.0180 0.3180 0.0210 0.0210 0.3210
0 0 0 1.0000 0 0
0 0 0 0 1.0000 0

A60 =
0.6300 0.0360 0.6360 0.0420 0.0420 0.6420
0.2100 0.0120 0.2120 0.0140 0.0140 0.2140
0 1.0000 0 0 0 0
0.2100 0.0120 0.2120 0.0140 0.0140 0.2140
0 0 0 1.0000 0 0
0 0 0 0 1.0000 0

There are several other correct answers. For example, some people decided to use a
state vector that included the past three samples of each of the $B_i$'s, i.e., a state of
dimension 9. Provided no errors were made, this works fine. At the other end, some
people found a state description that has a state dimension of 4.

b) **Asymptotic wealth growth rate.** This is going to depend on the eigenvalues of $A$, so first
we check the eigenvalues, which turn out to be

1.0627,  -0.3266 ± 0.4421i,  0,  0,  0

for $A_{35}$, and

1.0598,  -0.2019 ± 0.4015i,  0,  0,  0
for $A_{60}$. Each matrix has one real, positive eigenvalue that has larger magnitude than one, and than the other eigenvalues, which in fact all have magnitude less than one. Therefore, as $t \to \infty$,

$$A^t \to \lambda^t vw^T,$$

where $\lambda$ is the dominant eigenvalue, and $v$ and $w$ are the right and left eigenvectors associated with $\lambda$, normalized so that $w^Tv = 1$. There is a subtlety in computing $v$ and $w$, since the matrices have nontrivial Jordan form. Forming a matrix of right eigenvectors and inverting won’t work because there is no set of independent eigenvectors! But finding the eigenvector of $A^T$ associated with the dominant eigenvector, and then normalizing properly, does work. It follows that as $t \to \infty$ we have

$$x(t) \to \lambda^t vw^T x(0).$$

A quick check shows that $w^T x(0) \neq 0$ (which means the initial condition does excite the dominant mode), so as $t \to \infty$, $x(t)$ grows exponentially with rate $\lambda$. The wealth satisfies

$$w(t) \to \lambda^t (1^T v)(w^T x(0))$$

as $t \to \infty$. Hence $w(t + 1)/w(t)$ converges to $\lambda$. For the 35-35-30 scheme, the growth rate is 1.0627, i.e., 6.27% per year. For the 60-20-20 scheme, the growth rate is 1.0598, i.e., 5.98% per year. It makes sense that the first scheme has a higher growth rate since a higher fraction is invested in higher-yield CDs. A few people made numerical errors and ended up with growth rates of, say, 45%. How can investments in a portfolio of 5%, 6%, and 7% CDs end up yielding 45%? We can only assume this problem was solved very late at night ...

c) Asymptotic liquidity. The analysis above shows that as $t \to \infty$, the state vector grows exponentially with rate given by $\lambda$, and asymptotic distribution shape given by $v$, the right eigenvector associated with the dominant eigenvalue $\lambda$. Define

$$l_1 = [1 \ 0 \ 1 \ 0 \ 0 \ 1]^T, \quad l_2 = [0 \ 1 \ 0 \ 0 \ 1 \ 0]^T, \quad l_3 = [0 \ 0 \ 0 \ 1 \ 0 \ 0]^T,$$
so that $L_i(t) = l_i x(t)/1^T x(t)$. Since $x(t) \to \lambda^T w w^T x(0)$ we have $L_i(t) \to \frac{t^T v}{1^T v}$. Therefore the asymptotic values of $L_1$, $L_2$, and $L_3$ for the $35 – 35 – 30$ strategy are 0.5034, 0.3368, and 0.1598, respectively. For the $60 – 20 – 20$ strategy, they are 0.6215, 0.2499, and 0.1286, respectively. It makes sense that the second scheme is asymptotically more liquid, since a higher fraction is invested in fast-maturing CDs.

d) When we change $x(0)$ we affect the term $w^T x(0)$ in the asymptotic state. Hence by changing the initial condition we do not change the asymptotic growth rate. (Technically
there is one stupid case where we can change it: if we could arrange for \( w^T x(0) = 0 \), then asymptotic growth rate would be determined by the second largest eigenvalue. In fact for this problem you can prove that \( w^T x(0) = 0 \) is impossible since the vector \( w \) has all positive components. The asymptotic total wealth, which is \( \lambda^i(w^T x(0))(1^Tv) \), depends linearly on \( w^T x(0) \); evidently we want to choose \( x(0) \) to maximize \( w^T x(0) \). Thus our problem is to maximize

\[
  w^T[B_1(0) B_2(0) B_3(0) 0 0] = w_1B_1(0) + w_2B_2(0) + w_4B_3(0),
\]

subject to

\[
  B_1(0) + B_2(0) + B_3(0) = 1, \quad B_1(0) \geq 0, \quad B_2(0) \geq 0, \quad B_3(0) \geq 0.
\]

The solution is clear if you think about it: find the largest of \( w_1 \), \( w_2 \), and \( w_4 \) and allocate all the initial investment in the corresponding \( B_i \). We can obtain the dominant left eigenvector \( w \) in matlab, by finding the dominant right eigenvector of \( A^T \):

\[
  [W35, Lambda35] = eig(A35'); \quad \% \text{to find the left eigenvectors}
  w1 = W35(:,1) \quad \% \text{the dominant left eigenvector}
  w1 = w1/(w1'*v1_35) \quad \% \text{normalize w1 so that w1'*v=1}
\]

\[
\text{ans =}
0.4045
0.4074
0.4084
0.4174
0.4149
0.4122
\]

The optimal initial allocation will be \( B_1(0) = 0 \), \( B_2(0) = 0 \), and \( B_3(0) = 1 \). It’s just a bit better than the original scheme, since with original scheme \( w^T x(0) = 0.4094 \), whereas with our new optimal initial investment allocation we have \( w^T x(0) = 0.4174 \) (only 1.95% better). Plots below show the total wealth for the original allocation (solid line) and the optimal allocation (dashed line).

![Graph showing total wealth over time for original and optimal allocations](image)
Several people guessed this solution, but didn’t justify it. Others attempted SVD methods, or attempted to set up the initial condition along $v$, maybe by projecting $v$ onto the first, second, and fourth components. None of these methods is correct.