1. Harmonic oscillator. The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a harmonic oscillator.

a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

b) Sketch the vector field of the harmonic oscillator.

c) The state trajectories describe circular orbits, i.e., $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

2. Real modal form. We learned about the modal form of a system in class. Show that when some of eigenvalues of the dynamics matrix $A$ are complex, the system can be put in real modal form (Assuming the eigenvectors of $A$ are independent):

$$S^{-1}AS = \text{diag} \left( \Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1} \right)$$

where $\Lambda_r = \text{diag} (\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r+1, r+3, \ldots, n-1$$

where $\lambda_j$ are the complex eigenvalues (one from each conjugate pair). Clearly explain what the matrix $S$ is.

Generate a matrix $A$ in $\mathbb{R}^{10 \times 10}$ using $A=\text{randn}(10)$. (The entries of $A$ will be drawn from a unit normal distribution.) Find the eigenvalues of $A$. If by chance they are all real, generate a new instance of $A$. Find the real modal form of $A$, i.e., a matrix $S$ such that $S^{-1}AS$ has the real modal form. Your solution should include the source code that you use to find $S$, and some code that checks the results (i.e., computes $S^{-1}AS$ to verify it has the required form).


a) Show that $e^{A+B} = e^A e^B$ if $A$ and $B$ commute, i.e., $AB = BA$.

b) Carefully show that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$. 

4. Output response envelope for linear system with uncertain initial condition. We consider the autonomous linear dynamical system $\dot{x} = Ax$, $y(t) = Cx(t)$, where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$. We do not know the initial condition exactly; we only know that it lies in a ball of radius $r$ centered at the point $x_0$:

$$\|x(0) - x_0\| \leq r.$$ 

We call $x_0$ the nominal initial condition, and the resulting output, $y_{\text{nom}}(t) = Ce^{tA}x_0$, the nominal output. We define the maximum output or upper output envelope as

$$\bar{y}(t) = \max\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the maximum possible value of the output at time $t$, over all possible initial conditions. (Here you can choose a different initial condition for each $t$; you are not required to find a single initial condition.) In a similar way, we define the minimum output or lower output envelope as

$$\underline{y}(t) = \min\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the minimum possible value of the output at time $t$, over all possible initial conditions.

a) Explain how to find $\bar{y}(t)$ and $\underline{y}(t)$, given the problem data $A$, $C$, $x_0$, and $r$.

b) Carry out your method on the problem data in uie_data.m. On the same axes, plot $y_{\text{nom}}$, $\bar{y}$, and $\underline{y}$ versus $t$, over the range $0 \leq t \leq 10$.

5. Spectral mapping theorem. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is analytic, i.e., given by a power series expansion

$$f(u) = a_0 + a_1u + a_2u^2 + \cdots$$

(again, we’ll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbb{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if $\lambda$ is an eigenvalue of $A$, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the spectral mapping theorem.

To illustrate this with an example, generate a random $3 \times 3$ matrix, for example using $A = \text{randn}(3)$. Find the eigenvalues of $(I + A)(I - A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various computations.)

6. Interconnection of linear systems. Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are
themselves linear systems. In this problem you consider the specific interconnection shown below:

Here, there are two subsystems $S$ and $T$. Subsystem $S$ is characterized by

$$\dot{x} = Ax + B_1 u + B_2 w_1, \quad w_2 = Cx + D_1 u + D_2 w_1,$$

and subsystem $T$ is characterized by

$$\dot{z} = Fz + G_1 v + G_2 w_2, \quad w_1 = H_1 z, \quad y = H_2 z + Jw_2.$$

We don't specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (i.e., compatible) dimensions. Note that the subscripts in the matrices above, as in $B_1$ and $B_2$, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \quad \begin{bmatrix} x \\ z \end{bmatrix}, \quad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making.

7. **Analysis of investment allocation strategies.** Each year or period (denoted $t = 0, 1, \ldots$) an investor buys certain amounts of one-, two-, and three-year certificates of deposit (CDs) with interest rates 5%, 6%, and 7%, respectively. (We ignore minimum purchase requirements, and assume they can be bought in any amount.)

- $B_1(t)$ denotes the amount of one-year CDs bought at period $t$.
- $B_2(t)$ denotes the amount of two-year CDs bought at period $t$.
- $B_3(t)$ denotes the amount of three-year CDs bought at period $t$.

We assume that $B_1(0) + B_2(0) + B_3(0) = 1$, i.e., a total of 1 is to be invested at $t = 0$. (You can take $B_j(t)$ to be zero for $t < 0$.) The total payout to the investor, $p(t)$, at period $t$ is a sum of six terms:

- $1.05B_1(t-1)$, i.e., principle plus 5% interest on the amount of one-year CDs bought one year ago.
- $1.06B_2(t-2)$, i.e., principle plus 6% interest on the amount of two-year CDs bought two years ago.
• $1.07B_3(t - 3)$, i.e., principle plus 7% interest on the amount of three-year CDs bought three years ago.

• $0.06B_2(t - 1)$, i.e., 6% interest on the amount of two-year CDs bought one year ago.

• $0.07B_3(t - 1)$, i.e., 7% interest on the amount of three-year CDs bought one year ago.

• $0.07B_3(t - 2)$, i.e., 7% interest on the amount of three-year CDs bought two years ago.

The total wealth held by the investor at period $t$ is given by

$$w(t) = B_1(t) + B_2(t) + B_2(t - 1) + B_3(t) + B_3(t - 1) + B_3(t - 2).$$

Two re-investment allocation strategies are suggested.

• The 35-35-30 strategy. The total payout is re-invested 35% in one-year CDs, 35% in two-year CDs, and 30% in three-year CDs. The initial investment allocation is the same: $B_1(0) = 0.35$, $B_2(0) = 0.35$, and $B_3(0) = 0.30$.

• The 60-20-20 strategy. The total payout is re-invested 60% in one-year CDs, 20% in two-year CDs, and 20% in three-year CDs. The initial investment allocation is $B_1(0) = 0.60$, $B_2(0) = 0.20$, and $B_3(0) = 0.20$.

a) Describe the investments over time as a linear dynamical system $x(t + 1) = Ax(t)$, $y(t) = Cx(t)$ with $y(t)$ equal to the total wealth at time $t$. Be very clear about what the state $x(t)$ is, and what the matrices $A$ and $C$ are. You will have two such linear systems: one for the 35-35-30 strategy and one for the 60-20-20 strategy.

b) Asymptotic wealth growth rate. For each of the two strategies described above, determine the asymptotic growth rate, defined as $\lim_{t \to \infty} w(t+1)/w(t)$. (If this limit doesn’t exist, say so.) Note: simple numerical simulation of the strategies (e.g., plotting $w(t+1)/w(t)$ versus $t$ to guess its limit) is not acceptable. (You can, of course, simulate the strategies to check your answer.)

c) Asymptotic liquidity. The total wealth at time $t$ can be divided into three parts:

• $B_1(t) + B_2(t - 1) + B_3(t - 2)$ is the amount that matures in one year (i.e., the amount of principle we could get back next year)

• $B_2(t) + B_3(t - 1)$ is the amount that matures in two years

• $B_3(t)$ is the amount that matures in three years (i.e., is least liquid)

We define liquidity ratios as the ratio of these amounts to the total wealth:

$$L_1(t) = (B_1(t) + B_2(t - 1) + B_3(t - 2))/w(t),$$

$$L_2(t) = (B_2(t) + B_3(t - 1))/w(t),$$

$$L_3(t) = B_3(t)/w(t).$$

For the two strategies above, do the liquidity ratios converge as $t \to \infty$? If so, to what values? Note: as above, simple numerical simulation alone is not acceptable.
d) Suppose you could change the initial investment allocation for the 35-35-30 strategy, 
<i.e.,</i> choose some other nonnegative values for $B_1(0)$, $B_2(0)$, and $B_3(0)$ that satisfy 
$B_1(0) + B_2(0) + B_3(0) = 1$. What allocation would you pick, and how would it be 
better than the (0.35, 0.35, 0.30) initial allocation? (For example, would the asymptotic 
growth rate be larger?) How much better is your choice of initial investment allocations? 
<em>Hint for part d:</em> think very carefully about this one. <em>Hint for whole problem:</em> watch 
out for nondiagonalizable, or nearly nondiagonalizable, matrices. Don’t just blindly 
type in matlab commands; check to make sure you’re computing what you think you’re 
computing.