Question 6 is optional (it will not be graded) and you do not need to submit it to gradescope. However, doing them will help improve your understanding and grasp of the material.

1. Population dynamics. In this problem we will study how some population distribution (say, of people) evolves over time, using a discrete-time linear dynamical system model. Let $t = 0, 1, \ldots$ denote time in years (since the beginning of the study). The vector $x(t) \in \mathbb{R}^n$ will give the population distribution at year $t$ (on some fixed census date, e.g., January 1). Specifically, $x_i(t)$ is the number of people at year $t$, of age $i-1$. Thus $x_5(3)$ denotes the number of people of age 4, at year 3, and $x_1(t)$ (the number of 0 year-olds) denotes the number of people born since the last census. We assume $n$ is large enough that no one lives to age $n$.

We’ll also ignore the fact that $x_i$ are integers, and treat them as real numbers. (If $x_3(4) = 1.2$ bothers you, you can imagine the units as millions, say.) The total population at year $t$ is given by $\mathbf{1}^T x(t)$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector with all components 1.

- **Death rate.** The death rate depends only on age, and not on time $t$. The coefficient $d_i$ is the fraction of people of age $i-1$ who will die during the year. Thus we have, for $t = 0, 1, \ldots$, 
  $$x_{k+1}(t + 1) = (1 - d_k)x_k(t), \quad k = 1, \ldots, n - 1.$$  
  (As mentioned above, we assume that $d_n = 1$, i.e., all people who make it to age $n - 1$ die during the year.) The death rate coefficients satisfy $0 < d_i < 1$, $i = 1, \ldots, n - 1$. We define the survival rate coefficients as $s_k = 1 - d_k$, so $0 < s_k < 1$, $k = 1, \ldots, n - 1$.

- **Birth rate.** The birth rate depends only on age, and not on time $t$. The coefficient $b_i$ is the fraction of people of age $i-1$ who will have a child during the year (taking into account multiple births). Thus the total births during a year is given by
  $$x_1(t + 1) = b_1 x_1(t) + \cdots + b_n x_n(t).$$
  The birth rate coefficients satisfy $b_i \geq 0$, $i = 1, \ldots, n$. We’ll assume that at least one of the $b_k$’s is positive. (Of course you’d expect that $b_i$ would be zero for non-fertile ages, e.g., age below 11 and over 60, but we won’t make that explicit assumption.)

The assumptions imply the following important property of our model: if $x_i(0) > 0$ for $i = 1, \ldots, n$, then $x_i(t) > 0$ for $i = 1, \ldots, n$. Therefore we don’t have to worry about negative
\(x_i(t)\), so long as our initial population distribution \(x(0)\) has all positive components. (To use fancy language we’d say the system is \textit{positive orthant invariant}.)

a) Express the population dynamics model described above as a discrete-time linear dynamical system. That is, find a matrix \(A\) such that \(x(t + 1) = Ax(t)\).

b) Draw a block diagram of the system found in part (a).

c) Find the characteristic polynomial of the system explicitly in terms of the birth and death rate coefficients (or, if you prefer, the birth and survival rate coefficients).

d) \textit{Survival normalized variables}. For each person born, \(s_1\) make it to age 1, \(s_1s_2\) make it to age 2, and in general, \(s_1 \cdots s_k\) make it to age \(k\). We define

\[y_k(t) = \frac{x_k(t)}{s_1 \cdots s_{k-1}}\]

(with \(y_1(t) = x_1(t)\)) as new population variables that are normalized to the survival rate. Express the population dynamics as a linear dynamical system using the variable \(y(t) \in \mathbb{R}^n\). That is, find a matrix \(\tilde{A}\) such that \(\tilde{A}y(t + 1) = \tilde{A}y(t)\).

Determine whether each of the next four statements is true or false. (Of course by ‘true’ we mean true for any values of the coefficients consistent with our assumptions, and by ‘false’ we mean false for some choice of coefficients consistent with our assumptions.)

a) Let \(x\) and \(z\) both satisfy our population dynamics model, \(i.e., x(t + 1) = Ax(t)\) and \(z(t + 1) = Az(t)\), and assume that all components of \(x(0)\) and \(z(0)\) are positive. If \(1^T x(0) > 1^T z(0)\), then \(1^T x(t) > 1^T z(t)\) for \(t = 1, 2, \ldots\). (In words: we consider two populations that satisfy the same dynamics. Then the population that is initially larger will always be larger.)

b) All the eigenvalues of \(A\) are real.

c) If \(d_k \geq b_k\) for \(k = 1, \ldots, n\), then \(1^T x(t) \to 0\) as \(t \to \infty\), \(i.e.,\) the population goes extinct.

d) Suppose that \((b_1 + \cdots + b_n)/n \leq (d_1 + \cdots + d_n)/n\), \(i.e.,\) the ‘average’ birth rate is less than the ‘average’ death rate. Then \(1^T x(t) \to 0\) as \(t \to \infty\).

2. \textbf{Real modal form}. Generate a matrix \(A\) in \(\mathbb{R}^{10 \times 10}\) using \(A = \text{randn}(10)\). (The entries of \(A\) will be drawn from a unit normal distribution.) Find the eigenvalues of \(A\). If by chance they are all real, please generate a new instance of \(A\). Find the real modal form of \(A\), \(i.e.,\) a matrix \(S\) such that \(S^{-1}AS\) has the real modal form given in lecture 11. Your solution should include a clear explanation of how you will find \(S\), the source code that you use to find \(S\), and some code that checks the results (\(i.e.,\) computes \(S^{-1}AS\) to verify it has the required form).

3. \textbf{Minimum energy control}. Consider the discrete-time linear dynamical system

\[x(t + 1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots\]
where $x(t) \in \mathbb{R}^n$, and the input $u(t)$ is a scalar (hence, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$). The initial state is $x(0) = 0$.

a) Find the matrix $C_T$ such that

$$x(T) = C_T \begin{bmatrix} u(T - 1) \\
\vdots \\
u(1) \\
u(0) \end{bmatrix}.$$

b) For the remainder of this problem, we consider a specific system with $n = 4$. The dynamics and input matrices are

$$A = \begin{bmatrix}
0.5 & 0.7 & -0.9 & -0.5 \\
0.4 & -0.7 & 0.1 & 0.3 \\
0.7 & 0.0 & -0.6 & 0.1 \\
0.4 & -0.1 & 0.8 & -0.5 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}.$$ 

Suppose we want the state to be $x_{\text{des}}$ at time $T$. Consider the desired state

$$x_{\text{des}} = \begin{bmatrix}
0.8 \\
2.3 \\
-0.7 \\
-0.3 \\
\end{bmatrix}.$$ 

What is the smallest $T$ for which we can find inputs $u(0), \ldots, u(T - 1)$, such that $x(T) = x_{\text{des}}$? What are the corresponding inputs that achieve $x_{\text{des}}$ at this minimum time? What is the smallest $T$ for which we can find inputs $u(0), \ldots, u(T - 1)$, such that $x(T) = x_{\text{des}}$ for any $x_{\text{des}} \in \mathbb{R}^4$? We'll denote this $T$ by $T_{\text{min}}$.

c) Suppose the energy expended in applying inputs $u(0), \ldots, u(T - 1)$ is

$$E(T) = \sum_{t=0}^{T-1} (u(t))^2.$$ 

For a given $T$ (greater than $T_{\text{min}}$) and $x_{\text{des}}$, how can you compute the inputs which achieve $x(T) = x_{\text{des}}$ with the minimum expense of energy? Consider now the desired state

$$x_{\text{des}} = \begin{bmatrix}
-1 \\
1 \\
0 \\
1 \\
\end{bmatrix}.$$ 

For each $T$ ranging from $T_{\text{min}}$ to 30, find the minimum energy inputs that achieve $x(T) = x_{\text{des}}$. For each $T$, evaluate the corresponding input energy, which we denote by $E_{\text{min}}(T)$. Plot $E_{\text{min}}(T)$ as a function of $T$. (You should include in your solution a description of how you computed the minimum-energy inputs, and the plot of the minimum energy as a function of $T$. But you don’t need to list the actual inputs you computed!)
d) You should observe that $E_{\text{min}}(T)$ is non-increasing in $T$. Show that this is the case in general (i.e., for any $A$, $B$, and $x_{\text{des}}$).

Note: There is a direct way of computing the asymptotic limit of the minimum energy as $T \to \infty$. We'll cover these ideas in more detail in ee363.

4. Affine dynamical systems. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if it is a linear function plus a constant, i.e., of the form $f(x) = Ax + b$. Affine functions are more general than linear functions, which result when $b = 0$. We can generalize linear dynamical systems to affine dynamical systems, which have the form

\[ \dot{x} = Ax + Bu + f, \quad y = Cx + Du + g. \]

Fortunately we don’t need a whole new theory for (or course on) affine systems; a simple shift of coordinates converts it to a linear dynamical system. Assuming $A$ is invertible, define $\tilde{x} = x + A^{-1}f$ and $\tilde{y} = y - g + CA^{-1}f$. Show that $\tilde{x}$, $u$, and $\tilde{y}$ are the state, input, and output of a linear dynamical system.

5. Properties of symmetric matrices. In this problem $P$ and $Q$ are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

a) If $P \geq 0$ then $P + Q \geq Q$.

b) If $P \geq Q$ then $-P \leq -Q$.

c) If $P > 0$ then $P^{-1} > 0$.

d) If $P \geq Q > 0$ then $P^{-1} \leq Q^{-1}$.

e) If $P \geq Q$ then $P^2 \geq Q^2$.

Hint: you might find it useful for part (d) to prove $Z \geq I$ implies $Z^{-1} \leq I$.

6. Spectral mapping theorem. Suppose $f : \mathbb{R} \to \mathbb{R}$ is analytic, i.e., given by a power series expansion

\[ f(u) = a_0 + a_1u + a_2u^2 + \cdots \]

(where $a_i = f^{(i)}(0)/(i!)$). (You can assume that we only consider values of $u$ for which this series converges.) For $A \in \mathbb{R}^{n \times n}$, we define $f(A)$ as

\[ f(A) = a_0I + a_1A + a_2A^2 + \cdots \]

(again, we’ll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbb{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if $\lambda$ is an eigenvalue of $A$, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the spectral mapping theorem.

To illustrate this with an example, generate a random $3 \times 3$ matrix, for example using $A=\text{randn}(3)$. Find the eigenvalues of $(I + A)(I - A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various computations.)