1. Determining a linear system from experiments. Suppose $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$.
Two one-second experiments are performed. In the first, $x(0) = [1 \ 1]^T$ and $x(1) = [4 \ -2]^T$.
In the second, $x(0) = [1 \ 2]^T$ and $x(1) = [5 \ -2]^T$.

a) Find $x(1)$ and $x(2)$, given $x(0) = [3 \ -1]^T$.

b) Find $A$, by first computing the matrix exponential.

c) Either find $x(1.5)$ or explain why you cannot $(x(0) = [3 \ -1]^T)$.

d) More generally, for $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, describe a procedure for finding $A$ using experiments with different initial values. What conditions must be satisfied for your procedure to work?

2. Alignment of a fleet of vehicles. We consider a fleet of vehicles, labeled $1, \ldots, n$, which move along a line with (scalar) positions $y_1, \ldots, y_n$. We let $v_1, \ldots, v_n$ denote the velocities of the vehicles, and $u_1, \ldots, u_n$ the net forces applied to the vehicles. The vehicle motions are governed by the equations

$$\dot{y}_i = v_i, \quad \dot{v}_i = u_i - v_i.$$ 

(Here we take each vehicle mass to be one, and include a damping term in the equations.)

We assume that $y_1(0) < \cdots < y_n(0)$, i.e., the vehicles start out with vehicle 1 in the leftmost position, followed by vehicle 2 to its right, and so on, with vehicle $n$ in the rightmost position. The goal is for the vehicles to converge to the configuration

$$y_i = i, \quad v_i = 0, \quad i = 1, \ldots, n,$$

i.e., first vehicle at position 1, with unit spacing between adjacent vehicles, and all stationary.

We call this configuration aligned, and the goal is to drive the vehicles to this configuration, i.e., to align the vehicles. We define the spacing between vehicle $i$ and $i+1$ as $s_i(t) = y_{i+1}(t) - y_i(t)$, for $i = 1, \ldots, n - 1$. (When the vehicles are aligned, these spacings are all one.) We will investigate three control schemes for aligning the fleet of vehicles.

- Right looking control is based on the spacing to the vehicle to the right. We use the control law
  $$(u_i(t) = s_i(t) - 1, \quad i = 1, \ldots, n - 1,$$
for vehicles $i = 1, \ldots, n - 1$. In other words, we apply a force on vehicle $i$ proportional to its spacing error with respect to the vehicle to the right (i.e., vehicle $i + 1$). The rightmost vehicle uses the control law

$$u_n(t) = -(y_n(t) - n),$$

which applies a force proportional to its position error, in the opposite direction. This control law has the advantage that only the rightmost vehicle needs an absolute measurement sensor; the others only need a measurement of the distance to their righthand neighbor.

- **Left and right looking control** adjusts the input force based on the spacing errors to the vehicle to the left and the vehicle to the right:

$$u_i(t) = \frac{s_i(t) - 1}{2} - \frac{s_{i-1}(t) - 1}{2}, \quad i = 2, \ldots, n - 1,$$

The rightmost vehicle uses the same absolute error method as in right looking control, i.e.,

$$u_n(t) = -(y_n(t) - n),$$

and the first vehicle, which has no vehicle to its left, uses a right looking control scheme,

$$u_1(t) = s_1(t) - 1.$$

This scheme requires vehicle $n$ to have an absolute position sensor, but the other vehicles only need to measure the distance to their neighbors.

- **Independent alignment** is based on each vehicle independently adjusting its position with respect to its required position:

$$u_i(t) = -(y_i(t) - i), \quad i = 1, \ldots, n.$$

This scheme requires all vehicles to have absolute position sensors.

In the questions below, we consider the specific case with $n = 5$ vehicles.

a) Which of the three schemes work? By ‘work’ we mean that the vehicles converge to the alignment configuration, no matter what the initial positions and velocities are. Among the schemes that do work, which one gives the fastest asymptotic convergence to alignment? (If there is a tie between two or three schemes, say so.) In this part of the problem you can ignore the issue of vehicle collisions, i.e., spacings that pass through zero.

b) **Collisions.** In this problem we analyze vehicle collisions, which occur when any spacing between vehicles is equal to zero. (For example, $s_3(5.7) = 0$ means that vehicles 3 and 4 collide at $t = 5.7$.) We take the particular starting configuration

$$y = (0, 2, 3, 5, 7), \quad v = (0, 0, 0, 0, 0),$$

which corresponds to the vehicles with zero initial velocity, but not in the aligned positions. For each of the three schemes above (whether or not they work), determine if
a collision occurs. If a collision does occur, find the earliest collision, giving the time and the vehicles involved. (For example, ‘Vehicles 3 and 4 collide at $t = 7.7$.’) If there is a tie, i.e., two pairs of vehicles collide at the same time, say so. If the vehicles do not collide, find the point of closest approach, i.e., the minimum spacing that occurs, between any pair of vehicles, for $t \geq 0$. (Give the time, the vehicles involved, and the minimum spacing.) If there is a tie, i.e., two or more pairs of vehicles have the same distance of closest approach, say so. Be sure to give us times of collisions or closest approach with an absolute precision of at least 0.1.

3. Another formula for the matrix exponential. You might remember that for any complex number $a \in \mathbb{C}$, $e^a = \lim_{k \to \infty} (1 + a/k)^k$. You will establish the matrix analog: for any $A \in \mathbb{R}^{n \times n}$,

$$e^A = \lim_{k \to \infty} (I + A/k)^k.$$  

To simplify things, you can assume $A$ is diagonalizable. *Hint:* diagonalize.

4. Squareroot and logarithm of a (diagonalizable) matrix. Suppose that $A \in \mathbb{R}^{n \times n}$ is diagonalizable. Therefore, an invertible matrix $T \in \mathbb{C}^{n \times n}$ and diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ exist such that $A = T \Lambda T^{-1}$. Let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

a) We say $B \in \mathbb{R}^{n \times n}$ is a squareroot of $A$ if $B^2 = A$. Let $\mu_i$ satisfy $\mu_i^2 = \lambda_i$. Show that $B = T \text{diag}(\mu_1, \ldots, \mu_n)T^{-1}$ is a squareroot of $A$. A squareroot is sometimes denoted $A^{1/2}$ (but note that there are in general many squareroots of a matrix). When $\lambda_i$ are real and nonnegative, it is conventional to take $\mu_i = \sqrt{\lambda_i}$ (i.e., the nonnegative squareroot), so in this case $A^{1/2}$ is unambiguous.

b) (OPTIONAL) We say $B$ is a logarithm of $A$ if $e^B = A$, and we write $B = \log A$. Following the idea of part a, find an expression for a logarithm of $A$ (which you can assume is invertible). Is the logarithm unique? What if we insist on $B$ being real?

5. Minimum energy control. Consider the discrete-time linear dynamical system

$$x(t + 1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots$$

where $x(t) \in \mathbb{R}^n$, and the input $u(t)$ is a scalar (hence, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$). The initial state is $x(0) = 0$.

a) Find the matrix $C_T$ such that

$$x(T) = C_T \begin{bmatrix} u(T - 1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}.$$
b) For the remainder of this problem, we consider a specific system with $n = 4$. The dynamics and input matrices are

$$A = \begin{bmatrix} 0.5 & 0.7 & -0.9 & -0.5 \\ 0.4 & -0.7 & 0.1 & 0.3 \\ 0.7 & 0.0 & -0.6 & 0.1 \\ 0.4 & -0.1 & 0.8 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Suppose we want the state to be $x_{\text{des}}$ at time $T$. Consider the desired state $x_{\text{des}} = \begin{bmatrix} 0.8 \\ 2.3 \\ -0.7 \\ -0.3 \end{bmatrix}$.

What is the smallest $T$ for which we can find inputs $u(0), \ldots, u(T - 1)$, such that $x(T) = x_{\text{des}}$? What are the corresponding inputs that achieve $x_{\text{des}}$ at this minimum time? What is the smallest $T$ for which we can find inputs $u(0), \ldots, u(T - 1)$, such that $x(T) = x_{\text{des}}$ for any $x_{\text{des}} \in \mathbb{R}^4$? We’ll denote this $T$ by $T_{\text{min}}$.

c) Suppose the energy expended in applying inputs $u(0), \ldots, u(T - 1)$ is

$$E(T) = \sum_{t=0}^{T-1} (u(t))^2.$$

For a given $T$ (greater than $T_{\text{min}}$) and $x_{\text{des}}$, how can you compute the inputs which achieve $x(T) = x_{\text{des}}$ with the minimum expense of energy? Consider now the desired state $x_{\text{des}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

For each $T$ ranging from $T_{\text{min}}$ to 30, find the minimum energy inputs that achieve $x(T) = x_{\text{des}}$. For each $T$, evaluate the corresponding input energy, which we denote by $E_{\text{min}}(T)$. Plot $E_{\text{min}}(T)$ as a function of $T$. (You should include in your solution a description of how you computed the minimum-energy inputs, and the plot of the minimum energy as a function of $T$. But you don’t need to list the actual inputs you computed!)

d) You should observe that $E_{\text{min}}(T)$ is non-increasing in $T$. Show that this is the case in general (i.e., for any $A$, $B$, and $x_{\text{des}}$).

Note: There is a direct way of computing the asymptotic limit of the minimum energy as $T \to \infty$. We’ll cover these ideas in more detail in ee363.
6. Analysis of investment allocation strategies. Each year or period (denoted $t = 0, 1, \ldots$) an investor buys certain amounts of one-, two-, and three-year certificates of deposit (CDs) with interest rates 5%, 6%, and 7%, respectively. (We ignore minimum purchase requirements, and assume they can be bought in any amount.)

- $B_1(t)$ denotes the amount of one-year CDs bought at period $t$.
- $B_2(t)$ denotes the amount of two-year CDs bought at period $t$.
- $B_3(t)$ denotes the amount of three-year CDs bought at period $t$.

We assume that $B_1(0) + B_2(0) + B_3(0) = 1$, i.e., a total of 1 is to be invested at $t = 0$. (You can take $B_j(t)$ to be zero for $t < 0$.) The total payout to the investor, $p(t)$, at period $t$ is a sum of six terms:

- $1.05B_1(t-1)$, i.e., principle plus 5% interest on the amount of one-year CDs bought one year ago.
- $1.06B_2(t-2)$, i.e., principle plus 6% interest on the amount of two-year CDs bought two years ago.
- $1.07B_3(t-3)$, i.e., principle plus 7% interest on the amount of three-year CDs bought three years ago.
- $0.06B_2(t-1)$, i.e., 6% interest on the amount of two-year CDs bought one year ago.
- $0.07B_3(t-1)$, i.e., 7% interest on the amount of three-year CDs bought one year ago.
- $0.07B_3(t-2)$, i.e., 7% interest on the amount of three-year CDs bought two years ago.

The total wealth held by the investor at period $t$ is given by

$$w(t) = B_1(t) + B_2(t) + B_2(t-1) + B_3(t) + B_3(t-1) + B_3(t-2).$$

Two re-investment allocation strategies are suggested.

- **The 35-35-30 strategy.** The total payout is re-invested 35% in one-year CDs, 35% in two-year CDs, and 30% in three-year CDs. The initial investment allocation is the same: $B_1(0) = 0.35$, $B_2(0) = 0.35$, and $B_3(0) = 0.30$.

- **The 60-20-20 strategy.** The total payout is re-invested 60% in one-year CDs, 20% in two-year CDs, and 20% in three-year CDs. The initial investment allocation is $B_1(0) = 0.60$, $B_2(0) = 0.20$, and $B_3(0) = 0.20$.

a) Describe the investments over time as a linear dynamical system $x(t + 1) = Ax(t)$, $y(t) = Cx(t)$ with $y(t)$ equal to the total wealth at time $t$. Be very clear about what the state $x(t)$ is, and what the matrices $A$ and $C$ are. You will have two such linear systems: one for the 35-35-30 strategy and one for the 60-20-20 strategy.

b) **Asymptotic wealth growth rate.** For each of the two strategies described above, determine the asymptotic growth rate, defined as $\lim_{t \to \infty} w(t+1)/w(t)$. (If this limit doesn’t exist, say so.) Note: simple numerical simulation of the strategies (e.g., plotting $w(t+1)/w(t)$ versus $t$ to guess its limit) is not acceptable. (You can, of course, simulate the strategies to check your answer.)
c) **Asymptotic liquidity.** The total wealth at time $t$ can be divided into three parts:

- $B_1(t) + B_2(t - 1) + B_3(t - 2)$ is the amount that matures in one year (i.e., the amount of principle we could get back next year)
- $B_2(t) + B_3(t - 1)$ is the amount that matures in two years
- $B_3(t)$ is the amount that matures in three years (i.e., is least liquid)

We define liquidity ratios as the ratio of these amounts to the total wealth:

\[
L_1(t) = \frac{(B_1(t) + B_2(t - 1) + B_3(t - 2))}{w(t)},
\]
\[
L_2(t) = \frac{(B_2(t) + B_3(t - 1))}{w(t)},
\]
\[
L_3(t) = \frac{B_3(t)}{w(t)}.
\]

For the two strategies above, do the liquidity ratios converge as $t \to \infty$? If so, to what values? **Note:** as above, simple numerical simulation alone is not acceptable.

d) Suppose you could change the initial investment allocation for the 35-35-30 strategy, i.e., choose some other nonnegative values for $B_1(0), B_2(0),$ and $B_3(0)$ that satisfy $B_1(0) + B_2(0) + B_3(0) = 1$. What allocation would you pick, and how would it be better than the (0.35, 0.35, 0.30) initial allocation? (For example, would the asymptotic growth rate be larger?) How much better is your choice of initial investment allocations? **Hint for part d:** think very carefully about this one. **Hint for whole problem:** watch out for nondiagonalizable, or nearly nondiagonalizable, matrices. Don’t just blindly type in matlab commands; check to make sure you’re computing what you think you’re computing.