Homework 4 Solutions

EE 263 Stanford University Summer 2018

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1. Singularity of the KKT matrix. This problem concerns the general norm minimization with equality constraints problem (described in the lectures notes on pages 8-13),

\[
\begin{align*}
\text{minimize} & \quad \|Ax - b\| \\
\text{subject to} & \quad Cx = d
\end{align*}
\]

where the variable is \(x \in \mathbb{R}^n\), \(A \in \mathbb{R}^{m \times n}\), and \(C \in \mathbb{R}^{k \times n}\). We assume that \(C\) is fat (\(k \leq n\)), i.e., the number of equality constraints is no more than the number of variables.

Using Lagrange multipliers, we found that the solution can be obtained by solving the linear equations

\[
\begin{bmatrix}
A^T A & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\ \lambda
\end{bmatrix}
= \begin{bmatrix}
A^T b \\ d
\end{bmatrix}
\]

for \(x\) and \(\lambda\). (The vector \(x\) gives the solution of the norm minimization problem above.) The matrix above, which we will call \(K \in \mathbb{R}^{(n+k) \times (n+k)}\), is called the KKT matrix for the problem. (KKT are the initials of some of the people who came up with the optimality conditions for a more general type of problem.)

One question that arises is, when is the KKT matrix \(K\) nonsingular? The answer is: \(K\) is nonsingular if and only if \(C\) is full rank and \(\text{null}(A) \cap \text{null}(C) = \{0\}\).

You will fill in all details of the argument below.

a) Suppose \(C\) is not full rank. Show that \(K\) is singular.

b) Suppose that there is a nonzero \(u \in \text{null}(A) \cap \text{null}(C)\). Use this \(u\) to show that \(K\) is singular.

c) Suppose that \(K\) is singular, so there exists a nonzero vector \([u^T \ v^T]^T\) for which

\[
\begin{bmatrix}
A^T A & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
u \\ v
\end{bmatrix}
= 0.
\]

Write this out as two block equations, \(A^T Au + C^T v = 0\) and \(Cu = 0\). Conclude that \(u \in \text{null}(C)\). Multiply \(A^T Au + C^T v = 0\) on the left by \(u^T\), and use \(Cu = 0\) to conclude that \(\|Au\| = 0\), which implies \(u \in \text{null}(A)\). Finish the argument that leads to the conclusion that either \(C\) is not full rank, or \(\text{null}(A) \cap \text{null}(C) \neq \{0\}\).
Solution.

a) Suppose $C$ is not full rank. Since $C$ is fat, the rows of $C$ are linearly dependent. Therefore the last $k$ rows of the KKT matrix $[C \ 0]$ are not linearly independent and hence the KKT matrix is singular.

b) Suppose that there is a nonzero $u \in \text{null}(A) \cap \text{null}(C)$. Consider the vector

$$
\begin{bmatrix}
x \\
\lambda
\end{bmatrix} = \begin{bmatrix}
u \\
0
\end{bmatrix}.
$$

This nonzero vector is the nullspace of the KKT matrix. Therefore the KKT matrix is singular.

c) Suppose that $K$ is singular. So there exists a nonzero vector $[u^T \ v^T]^T$ for which

$$
\begin{bmatrix}
A^T A & C^T \\
C & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix} = 0,
$$

which is equivalently written as $A^T Au + C^T v = 0$ and $Cu = 0$. Thus $u \in \text{null}(C)$. Multiplying $A^T Au + C^T v = 0$ on the left by $u^T$, and using $Cu = 0$ gives $\|Au\| = 0$, which implies $u \in \text{null}(A)$. If $u \neq 0$ then we have a nonzero element in $\text{null}(C)$ and $\text{null}(A)$, and therefore in $\text{null}(C) \cap \text{null}(A)$. Thus $\text{null}(C) \cap \text{null}(A) \neq \{0\}$. Otherwise, if $u = 0$, then $v \neq 0$ and we have $C^T v = 0$. Therefore $C$ is not full rank.

This implies that the KKT matrix $K$ is nonsingular if and only if $C$ is full rank and $\text{null}(A) \cap \text{null}(C) = \{0\}$.

2. Analysis of a power control algorithm. In this problem we consider again the power control method described in homework problem 2.1 Please refer to this problem for the setup and background. In that problem, you expressed the power control method as a discrete-time linear dynamical system, and simulated it for a specific set of parameters, with several values of initial power levels, and two target SINRs. You found that for the target SINR value $\gamma = 3$, the powers converged to values for which each SINR exceeded $\gamma$, no matter what the initial powers was, whereas for the larger target SINR value $\gamma = 5$, the powers appeared to diverge, and the SINRs did not appear to converge. You are going to analyze this, now that you know alot more about linear systems.

a) Explain the simulations. Explain your simulation results from the problem 1(b) for the given values of $G$, $\alpha$, $\sigma$, and the two SINR threshold levels $\gamma = 3$ and $\gamma = 5$.

b) Critical SINR threshold level. Let us consider fixed values of $G$, $\alpha$, and $\sigma$. It turns out that the power control algorithm works provided the SINR threshold $\gamma$ is less than some critical value $\gamma_{\text{crit}}$ (which might depend on $G$, $\alpha$, $\sigma$), and doesn’t work for $\gamma > \gamma_{\text{crit}}$. (‘Works’ means that no matter what the initial powers are, they converge to values for which each SINR exceeds $\gamma$.) Find an expression for $\gamma_{\text{crit}}$ in terms of $G \in \mathbb{R}^{n \times n}$, $\alpha$, and $\sigma$. Give the simplest expression you can. Of course you must explain how you came up with your expression.
Solution.

a) In the homework we found that the powers propagate according to a linear system. The power update rule for a single transmitter can be found by manipulating the definitions given in the problem.

\[
p_i(t+1) = \frac{\alpha \gamma p_i(t)}{S_i(t)} = \frac{\alpha \gamma p_i(t)q_i(t)}{s_i(t)} = \frac{\alpha \gamma p_i(t) \left[ \sigma + \sum_{j \neq i} G_{ij} p_j(t) \right]}{G_{ii} p_i(t)}
\]

In matrix form the equations represent a linear dynamical system with constant input, \( p(t+1) = Ap(t) + b \).

\[
\begin{bmatrix}
p_1(t+1) \\
p_2(t+1) \\
p_3(t+1) \\
\vdots \\
p_n(t+1)
\end{bmatrix} = \alpha \gamma
\begin{bmatrix}
0 & G_{12} & G_{13} & \cdots & G_{1n} \\
G_{21} & 0 & G_{23} & \cdots & G_{2n} \\
G_{31} & G_{32} & 0 & \cdots & G_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G_{n1} & G_{n2} & G_{n3} & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\
p_2(t) \\
p_3(t) \\
\vdots \\
p_n(t)
\end{bmatrix} + \alpha \gamma
\begin{bmatrix}
\sigma \\
\sigma \\
\sigma \\
\vdots \\
\sigma
\end{bmatrix}
\begin{bmatrix}
p(t) \\
\vdots \\
\vdots \\
\vdots \\
b
\end{bmatrix}
\]

where \( A = \alpha \gamma P \). This is a discrete LDS, and is stable if and only if \( |\lambda_i| < 1 \) for all \( i = 1, \ldots, n \), where \( \lambda_i \) are the eigenvalues of \( A \). When \( \gamma = 3 \) the eigenvalues of \( A \) are 0.6085, \(-0.3600\), and \(-0.2485\), so the system is stable; for all initial conditions, the powers converge to their equilibrium values.

Also, the SINR at each receiver \( i \), given by \( S_i \), converges to the same constant value \( \alpha \gamma \), which is enough for a successful signal reception. This can be shown by observing that at equilibrium \( p_i(t+1) = p_i(t) = \bar{p}_i \), and the power update equation gives

\[
\bar{p}_i = \bar{p}_i (\alpha \gamma / S_i(t)).
\]

After cancellation, we obtain the constant value for each SINR, \( S_i = \alpha \gamma \).

When \( \gamma = 5 \), the eigenvalues of \( A \) are 1.0141, \(-0.6000\), and \(-0.4141\). This system is unstable because of the first eigenvalue, so this means there are initial conditions from which the powers diverge.

\[
\gg \text{inv(v)*b}
\]

\[-0.0670\]
\[-0.0000\]
\[-0.0182\]

b) The critical SINR threshold level is a function of dominant system eigenvalue. We will assume that matrix \( P \) is diagonalizable and that its eigenvalues are ordered by their
magnitude when forming $\Lambda$ matrix. Using the property that scaling of any matrix scales its eigenvalues by the same constant, we can derive:

$$A = \alpha \gamma P = \alpha \gamma T \Lambda T^{-1} = T \text{diag}(\alpha \gamma \lambda_1, \ldots, \alpha \gamma \lambda_n) T^{-1}$$

For a marginally stable system we need to have $|\alpha \gamma \lambda_1| \leq 1$. Manipulating equation $\alpha \gamma \text{crit} |\lambda_1| = 1$, we obtain the critical SINR threshold level,

$$\gamma_{\text{crit}} = \frac{1}{\alpha |\lambda_1|}.$$ 

3. **Linear system with one-bit quantized output.** We consider the system

$$\dot{x} = Ax, \quad y(t) = \text{sign}(cx(t))$$

where

$$A = \begin{bmatrix} -0.1 & 1 \\ -1 & 0.1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

and the sign function is defined as

$$\text{sign}(a) = \begin{cases} +1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Roughly speaking, the output of this autonomous linear system is quantized to one-bit precision. The following outputs are observed:

$$y(0.4) = +1, \quad y(1.2) = -1, \quad y(2.3) = -1, \quad y(3.8) = +1$$

What can you say (if anything) about the following:

$$y(0.7), \quad y(1.8), \quad \text{and } y(3.7)?$$

Your response might be, for example: “$y(0.7)$ is definitely +1, and $y(1.8)$ is definitely −1, but $y(3.7)$ can be anything (i.e., −1, 0, or 1)”.

Of course you must fully explain how you arrive at your conclusions. (What we mean by “$y(0.7)$ is definitely +1” is: for any trajectory of the system for which $y(0.4) = +1$, $y(1.2) = -1$, $y(2.3) = -1$, and $y(3.8) = +1$, we also have $y(0.7) = +1$.)

**Solution.** We know that $x(t) = e^{At}x(0)$, so we have

$$y(t) = \text{sign}(cx(t)) = \text{sign}(ce^{At}x(0)).$$

What’s unknown here is the exact value of $x(0)$. The given output data is

$$ce^{A0.4}x(0) > 0, \quad ce^{A1.2}x(0) < 0, \quad ce^{A2.3}x(0) < 0, \quad ce^{A3.8}x(0) > 0.$$
These data give us information about $x(0)$. From that information about $x(0)$, we can (maybe) determine information about

$$y(0.7) = \text{sign} \left( ce^{A_{0.7}x(0)} \right), \quad y(1.8) = \text{sign} \left( ce^{A_{1.8}x(0)} \right), \quad y(3.7) = \text{sign} \left( ce^{A_{3.7}x(0)} \right).$$

Geometrically, a constraint of the form $fx(0) > 0$, where $f \in \mathbb{R}^{1 \times 2}$ is a row vector, defines a halfspace in $\mathbb{R}^2$, with inward normal vector $f^T$. Therefore each of our four measurements of $y$ gives us a halfspace in $\mathbb{R}^2$ that $x(0)$ must lie in. Taking the intersection, we see that $x(0)$ must lie in a cone or sector in $\mathbb{R}^2$. This cone is the set of all possible values of $x(0)$ that are consistent with the measured data. This is shown below.
To determine the sign of $y(t)$ for $t = 0.7$, 1.8, and 3.7, we have to check the sign of $ce^{At}x(0)$, for $t = 0.7$, 1.8, 3.7. Geometrically, this means we are checking whether the cone of possible values of $x(0)$ lies on one side, or both sides, of the halfspaces determine by $ce^{At}$, for $t = 0.7$, 1.8, 3.7. This results in $y(0.7) = -1$, and $y(1.8) = -1$, whereas $y(3.7)$ cannot be determined, since the
boundary of the halfplanes lies within the uncertainty region.

The matlab code below solves the problem using the approach just described.

```matlab
A = [-0.1 1; -1 0.1]; c = [ 1 -1];
M_1 = c*expm(A*0.4); M_2 = c*expm(A*1.2); M_3 = c*expm(A*2.3); M_4 = c*expm(A*3.8);
x = linspace(-10,10); coeff1 = -M_1(1)/M_1(2); y1 = coeff1*x;
coeff2 = -M_2(1)/M_2(2); y2 = coeff2*x; coeff3 = -M_3(1)/M_3(2);
y3 = coeff3*x; coeff4 = -M_4(1)/M_4(2); y4 = coeff4*x;
M_5 = c*expm(A*0.7); coeff5 = -M_5(1)/M_5(2); y5 = coeff5*x; M_6 = c*expm(A*1.8);
coeff6 = -M_6(1)/M_6(2); y6 = coeff6*x; M_7 = c*expm(A*3.7); coeff7 = -M_7(1)/M_7(2); y7 = coeff7*x;
[a,b] = meshgrid(-10:.5:10,-10:.5:10); figure(1); clf;
subplot(2,2,1); plot(x,y1); grid on; hold on;
for i=1:(size(a,1))^2;
    if (b(i)<coeff1*a(i)); plot(a(i),b(i),'.'); end;
end;
title('Halfplanes for y(0.4)');
subplot(2,2,2); plot(x,y1); grid on; hold on;
for i=1:(size(a,1))^2;
    if (b(i)<coeff1*a(i)); plot(a(i),b(i),'.'); end;
end;
title('Halfplanes for y(1.8)');
subplot(2,2,3); plot(a(i),b(i),'.'); end;
end;
title('Halfplanes for y(3.7)');
```

```matlab
subplot(2,2,3); plot(a(i),b(i),'.'); end;
end;
```
plot(x,y2,'--'); for i=1:(size(a,1))^2; if (b(i)<coeff2*a(i)); plot(a(i),b(i),'+'); end; end; title('Adding y(1.2)'); subplot(2,2,3); plot(x(1:50),y1(1:50)); grid on; hold on; axis([-10 10 -10 10]); for i=1:(size(a,1))^2; if ((b(i)<coeff1*a(i)) & (b(i)<coeff2*a(i))); plot(a(i),b(i),'.'); end; if (b(i)<coeff3*a(i)); plot(a(i),b(i),'+'); end; end; plot(x(51:100),y2(51:100)); plot(x,y3,'--'); title('Adding y(2.3)'); subplot(2,2,4); plot(x(1:50),y1(1:50)); grid on; hold on; axis([-10 10 -10 10]); for i=1:(size(a,1))^2; if ((b(i)<coeff1*a(i)) & (b(i)<coeff2*a(i))); plot(a(i),b(i),'.'); end; if (b(i)>coeff4*a(i)); plot(a(i),b(i),'+'); end; end; plot(x(51:100),y2(51:100)); plot(x,y4,'--'); title('Adding y(3.8)'); figure(2); clf plot(x(1:50),y1(1:50)); grid on; hold on; axis([-10 10 -10 10]); for i=1:(size(a,1))^2; if ((b(i)<coeff1*a(i)) & (b(i)>coeff4*a(i))); plot(a(i),b(i),'.'); end; if (b(i)<coeff5*a(i)); plot(a(i),b(i),'+'); end; end; plot(x(1:50),y4(1:50)); plot(x(1:50),y4(1:50)); plot(x,y5,'--'); title('Resulting uncertainty region for x(0)'); figure(3); clf subplot(2,2,1); plot(x(1:50),y1(1:50)); grid on; hold on; axis([-10 10 -10 10]); for i=1:(size(a,1))^2; if ((b(i)<coeff1*a(i)) & (b(i)>coeff4*a(i))); plot(a(i),b(i),'.'); end; if (b(i)<coeff5*a(i)); plot(a(i),b(i),'+'); end; end; plot(x(1:50),y4(1:50)); plot(x,y5,'--');
Some of you used another method, which was less geometric but perfectly correct. First we observe that the eigenvalues are $\pm 0.995j$, so all solutions are periodic, and we have

$$ ce^{At}x(0) = \alpha \cos(0.995t + \theta), $$

where $\alpha \geq 0$ and $\theta$ give another parametrization of all possible trajectories. In this problem all that matters in the sign of this signal, which simply changes every $\pi/0.995 = 3.16$ seconds. What we need to do is figure out bounds on when the signal zero crossings can be. Evidently the positive to negative crossing takes place between $t = 0.4$ and $t = 1.2$, and the negative to positive transition occurs between $2.3$ and $3.6$. Putting this together, using the fact that the two crossings are $3.16$ seconds apart, we find the original crossing occurs between $0.4 + 3.16/2 = 0.64$. Hence the negative to positive crossing occurs between $0.4 + 3.16 = 3.56$ and $3.8$. From these fact we see that $y(0.7) = -1$, $y(1.8) = -1$, whereas $y(3.7)$ cannot be determined.

4. **Some basic properties of eigenvalues.** Show the following:

   a) The eigenvalues of $A$ and $A^T$ are the same.

   b) $A$ is invertible if and only if $A$ does not have a zero eigenvalue.

   c) If the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ and $A$ is invertible, then the eigenvalues of $A^{-1}$ are $1/\lambda_1, \ldots, 1/\lambda_n$.

   d) The eigenvalues of $A$ and $T^{-1}AT$ are the same.
5. Tax policies. In this problem we explore a dynamic model of an economy, including the effects of government taxes and spending, which we assume (for simplicity) takes place at the beginning of each year. Let \( x(t) \in \mathbb{R}^n \) represent the pre-tax economic activity at the beginning of year \( t \), across \( n \) sectors, with \( x(t)_i \) being the pre-tax activity level in sector \( i \). We let \( \tilde{x}(t) \in \mathbb{R}^n \) denote the post-tax economic activity, across \( n \) sectors, at the beginning of year \( t \). We will assume that all entries of \( x(0) \) are positive, which will imply that all entries of \( x(t) \) and \( \tilde{x}(t) \) are positive, for all \( t \geq 0 \).

The pre- and post-tax activity levels are related as follows. The government taxes the sector activities at rates given by \( r \in \mathbb{R}^n \), with \( r_i \) the tax rate for sector \( i \). These rates all satisfy \( 0 \leq r_i < 1 \). The total government revenue is then \( R(t) = r^T x(t) \). This total revenue is then spent in the sectors proportionally, with \( s \in \mathbb{R}^n \) giving the spending proportions in the sectors. These spending proportions satisfy \( s_i \geq 0 \) and \( \sum_{i=1}^{n} s_i = 1 \); the spending in sector \( i \) is \( s_i R(t) \). The post-tax economic activity in sector \( i \), which accounts for the government taxes and spending, is then given by

\[
\tilde{x}(t)_i = x(t)_i - r_i x(t)_i + s_i R(t), \quad i = 1, \ldots, n, \quad t = 0, 1, \ldots
\]

**Hint:** you’ll need to use the facts that \( \det A = \det(A^T) \), \( \det(AB) = \det A \det B \), and, if \( A \) is invertible, \( \det A^{-1} = 1/\det A \).

**Solution.**

a) The eigenvalues of a matrix \( A \) are given by the roots of the polynomial \( \det(sI - A) \).
From determinant properties we know that \( \det(sI - A) = \det(sI - A)^T = \det(sI - A^T) \).
We conclude that the eigenvalues of \( A \) and \( A^T \) are the same.

b) First we recall that \( A \) is invertible if and only if \( \det(A) \neq 0 \). But \( \det(A) \neq 0 \iff \det(-A) \neq 0 \).

i. If 0 is an eigenvalue of \( A \), then \( \det(sI - A) = 0 \) when \( s = 0 \). It follows that \( \det(-A) = 0 \) and thus \( \det(A) = 0 \), and \( A \) is not invertible. From this fact we conclude that if \( A \) is invertible, then 0 is not an eigenvalue of \( A \).

ii. If \( A \) is not invertible, then \( \det(A) = \det(-A) = 0 \). This means that, for \( s = 0 \), \( \det(sI - A) = 0 \), and we conclude that in this case 0 must be an eigenvalue of \( A \). From this fact it follows that if 0 is not an eigenvalue of \( A \), then \( A \) is invertible.

c) From the results of the last item we see that 0 is not an eigenvalue of \( A \). Now consider the eigenvalue/eigenvector pair \( (\lambda_i, x_i) \) of \( A \). This pair satisfies \( Ax_i = \lambda_i x_i \). Now, since \( A \) is invertible, \( \lambda_i \) is invertible. Multiplying both sides by \( A^{-1} \) and multiplying both sides by \( A^{-1} \) we have \( \lambda_i^{-1} x_i = A^{-1} x_i \), and from this we conclude that the eigenvalues of the inverse are the inverse of the eigenvalues.

d) First we note that \( \det(sI - A) = \det(I(sI - A)) = \det(T^{-1} T(sI - A)) \). Now, from determinant properties, we have \( \det(T^{-1} T(sI - A)) = \det(T^{-1} (sI - A) T) \). But this is also equal to \( \det(sI - T^{-1} AT) \), and the conclusion is that the eigenvalues of \( A \) and \( T^{-1} AT \) are the same.
Economic activity propagates from year to year as \( x(t + 1) = E \tilde{x}(t) \), where \( E \in \mathbb{R}^{n \times n} \) is the input-output matrix of the economy. You can assume that all entries of \( E \) are positive.

We let \( S(t) = \sum_{i=1}^{n} x(t)_i \) denote the total economic activity in year \( t \), and we let

\[
G = \lim_{t \to \infty} \frac{S(t + 1)}{S(t)}
\]

denote the (asymptotic) growth rate (assuming it exceeds one) of the economy.

a) Explain why the growth rate does not depend on \( x(0) \) (unless it exactly satisfies a single linear equation, which we rule out as essentially impossible). Express the growth rate \( G \) in terms of the problem data \( r, s, \) and \( E \), using ideas from the course. You may assume that a matrix that arises in your analysis is diagonalizable and has a single dominant eigenvalue, i.e., an eigenvalue \( \lambda_1 \) that satisfies \(|\lambda_1| > |\lambda_i|\) for \( i = 2, \ldots, n \). (These assumptions aren’t actually needed—they’re just to simplify the problem for you.)

b) Consider the problem instance with data

\[
E = \begin{bmatrix}
0.3 & 0.4 & 0.1 & 0.6 \\
0.2 & 0.3 & 0.7 & 0.2 \\
0.1 & 0.2 & 0.2 & 0.1 \\
0.4 & 0.2 & 0.3 & 0.2
\end{bmatrix}, \quad r = \begin{bmatrix}
0.45 \\
0.25 \\
0.1 \\
0.1
\end{bmatrix}, \quad s = \begin{bmatrix}
0.15 \\
0.3 \\
0.4 \\
0.15
\end{bmatrix}.
\]

Find the growth rate. Now find the growth rate with the tax rate set to zero, i.e., \( r = 0 \) (in which case \( s \) doesn’t matter). You are welcome (even, encouraged) to simulate the economic activity to double-check your answer, but we want the value using the expression found in part (a).

**Solution.** Tracing through the equations, we get \( x(t + 1) = A x(t) \), where

\[
A = E(I - \text{diag}(r) + s r^T).
\]

Alternatively, we can express \( A \) by its entries as

\[
A_{ij} = E_{ij}(1 - r_j) + (Es)_i r_j, \quad i, j = 1, \ldots, n.
\]

It follows that \( x(t) = A^t x(0) \).

Using our assumption that \( A \) is diagonalizable, we have

\[
x(t) = \sum_{i=1}^{n} \lambda_i^t v_i w_i^T x(0),
\]

where \( \lambda_i \) are the eigenvalues of \( A \), \( v_i \) the associated eigenvectors, and \( w_i \) the associated left eigenvectors (suitably normalized). We sort the eigenvalues so \( \lambda_1 \) is dominant: \(|\lambda_1| > |\lambda_i|\) for \( i = 2, \ldots, n \). Then assuming that \( w_1^T x(0) \neq 0 \), we can approximate \( x(t) \) as

\[
x(t) \approx \lambda_1^t v_1(w_1^T x(0)).
\]
Now we see that $\lambda_1$ must be positive; if not, then the equation above shows $x(t)$ would eventually have negative entries, which is impossible. (The same argument shows that the entries of $v_1$ must be nonnegative.) Thus we have

$$S(t) \approx \lambda_1^t (1^T v_1) (w_1^T x(0)).$$

(Note that since the entries of $v_1$ are nonnegative, we cannot have $1^T v_1 = 0$; if that were true, then $v_1 = 0$.) Thus, we have

$$S(t + 1)/S(t) \approx \lambda_1.$$

So $G = \lambda_1$.

Now evaluating $G$ for the given data, we find that $G = 1.0908$ (9.08\% growth) without taxes, and $G = 1.1237$ (12.37\%) with taxes. So, the taxes have actually boosted the asymptotic economic growth rate! Tell that to the next person who tells you that taxes are always bad (if you know any such people).

A script for computing the growth rates is given below.

```matlab
E = [0.3 0.4 0.1 0.6;
     0.2 0.3 0.7 0.2;
     0.1 0.2 0.2 0.1;
     0.4 0.2 0.3 0.2];

r = [0.45; 0.25; 0.1; 0.1];
s = [0.15; 0.3; 0.4; 0.15];

G_no_tax = max(abs(eig(E)));
A = E*(eye(n)-diag(r)+s*r');
G_with_tax = max(abs(eig(A)));
```