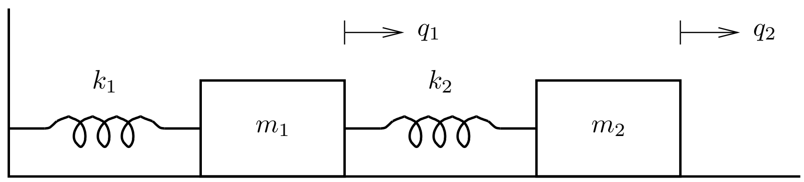
Homework 4

July 25, 2017

1. **Controlling a system using the initial conditions.** Consider the mechanical system shown below:



Here q_i give the displacements of the masses, m_i are the values of the masses, and k_i are the spring stiffnesses, respectively. The dynamics of this system are

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix} x$$

where the state is given by

$$x = \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}.$$

Immediately before $t = 0$, you are able to apply a strong impulsive force α_i to mass i, which results in initial condition

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ \alpha_1/m_1 \\ \alpha_2/m_2 \end{bmatrix}.$$

(i.e., each mass starts with zero position and a velocity determined by the impulsive forces.) This problem concerns selection of the impulsive forces α_1 and α_2. For parts a–c below, the parameter values are

$$m_1 = m_2 = 1, \quad k_1 = k_2 = 1.$$

Consider the following specifications:

a) $q_2(10) = 2$

b) $q_1(10) = 1$, $q_2(10) = 2$

c) $q_1(10) = 1$, $q_2(10) = 2$, $\dot{q}_1(10) = 0$, $\dot{q}_2(10) = 0$

d) $q_2(10) = 2$ when the parameters have the values used above (*i.e.*, $m_1 = m_2 = 1$, $k_1 = k_2 = 1$), and *also*, $q_2(10) = 2$ when the parameters have the values $m_1 = 1$, $m_2 = 1.3$, $k_1 = k_2 = 1$.

Determine whether each of these specifications is feasible or not (*i.e.*, whether there exist $\alpha_1,\ \alpha_2 \in \mathbb{R}$ that make the specification hold). If the specification is feasible, find the particular $\alpha_1,\ \alpha_2$ that satisfy the specification and minimize $\alpha_1^2+\alpha_2^2$. If the specification is infeasible, find the particular $\alpha_1,\ \alpha_2$ that come closest, in a least-squares sense, to satisfying the specification. (For example, if you cannot find $\alpha_1,\ \alpha_2$ that satisfy $q_1(10) = 1$, $q_2(10) = 2$, then find α_i that minimize $(q_1(10)-1)^2+(q_2(10)-2)^2$.) Be sure to be very clear about which alternative holds for each specification.

2. **Analysis of a power control algorithm.** In this problem we consider again the power control method described in homework problem 2.1 Please refer to this problem for the setup and background. In that problem, you expressed the power control method as a discrete-time linear dynamical system, and simulated it for a specific set of parameters, with several values of initial power levels, and two target SINRs. You found that for the target SINR value $\gamma = 3$, the powers converged to values for which each SINR exceeded γ, no matter what the initial power was, whereas for the larger target SINR value $\gamma = 5$, the powers appeared to diverge, and the SINRs did not appear to converge. You are going to analyze this, now that you know alot more about linear systems.

   a) *Explain the simulations.* Explain your simulation results from the problem 1(b) for the given values of G, α, σ, and the two SINR threshold levels $\gamma = 3$ and $\gamma = 5$.

   b) *Critical SINR threshold level.* Let us consider fixed values of G, α, and σ. It turns out that the power control algorithm works provided the SINR threshold γ is less than some critical value γ_{crit} (which might depend on G, α, σ), and doesn't work for $\gamma > \gamma_{\text{crit}}$. ('Works' means that no matter what the initial powers are, they converge to values for which each SINR exceeds γ.) Find an expression for γ_{crit} in terms of $G \in \mathbb{R}^{n\times n}$, α, and σ. Give the simplest expression you can. Of course you must explain how you came up with your expression.

3. **Harmonic oscillator.** The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a *harmonic oscillator.*

   a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

   b) Sketch the vector field of the harmonic oscillator.

   c) The state trajectories describe circular orbits, *i.e.*, $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

   d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

4. **Norm expressions for quadratic forms.** Let $f(x) = x^\mathsf{T} Ax$ (with $A = A^\mathsf{T} \in \mathbb{R}^{n\times n}$) be a quadratic form.

   a) Show that f is positive semidefinite (*i.e.*, $A \geq 0$) if and only if it can be expressed as $f(x) = \|Fx\|^2$ for some matrix $F \in \mathbb{R}^{k\times n}$. Explain how to find such an F (when $A \geq 0$). What is the size of the smallest such F (*i.e.*, how small can k be)?

   b) Show that f can be expressed as a difference of squared norms, in the form $f(x) = \|Fx\|^2 - \|Gx\|^2$, for some appropriate matrices F and G. How small can the sizes of F and G be?

5. **Properties of symmetric matrices.** In this problem P and Q are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

   a) If $P \geq 0$ then $P + Q \geq Q$.

   b) If $P \geq Q$ then $-P \leq -Q$.

   c) If $P > 0$ then $P^{-1} > 0$.

   d) If $P \geq Q > 0$ then $P^{-1} \leq Q^{-1}$.

   e) If $P \geq Q$ then $P^2 \geq Q^2$.

   Hint: you might find it useful for part (d) to prove $Z \geq I$ implies $Z^{-1} \leq I$.

6. **Frobenius norm of a matrix.** The Frobenius norm of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\|A\|_\mathrm{F} = \sqrt{\operatorname{trace} A^\mathsf{T} A}$. (Recall trace is the trace of a matrix, *i.e.*, the sum of the diagonal entries.)

   a) Show that
   $$\|A\|_\mathrm{F} = \left(\sum_{i,j} |A_{ij}|^2 \right)^{1/2}.$$
   Thus the Frobenius norm is simply the Euclidean norm of the matrix when it is considered as an element of $\mathbb{R}^{n^2}$. Note also that it is much easier to compute the Frobenius norm of a matrix than the (spectral) norm (*i.e.*, maximum singular value).

   b) Show that if U and V are orthogonal, then $\|UA\|_\mathrm{F} = \|AV\|_\mathrm{F} = \|A\|_\mathrm{F}$. Thus the Frobenius norm is not changed by a pre- or post- orthogonal transformation.

   c) Show that $\|A\|_\mathrm{F} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where $\sigma_1, \ldots, \sigma_r$ are the singular values of A. Then show that $\sigma_{\max}(A) \leq \|A\|_\mathrm{F} \leq \sqrt{r}\sigma_{\max}(A)$. In particular, $\|Ax\| \leq \|A\|_\mathrm{F}\|x\|$ for all x.

7. **Drawing a graph.** We consider the problem of drawing (in two dimensions) a graph with n vertices (or nodes) and m undirected edges (or links). This just means assigning an x- and a y- coordinate to each node. We let $x \in \mathbb{R}^n$ be the vector of x- coordinates of the nodes, and $y \in \mathbb{R}^n$ be the vector of y- coordinates of the nodes. When we draw the graph, we draw node i at the location $(x_i, y_i) \in \mathbb{R}^2$. The problem, of course, is to make the drawn graph look good. One goal is that neighboring nodes on the graph (*i.e.*, ones connected by an edge) should not be too far apart as drawn. To take this into account, we will choose the x- and y-coordinates so as to minimize the objective

$$J = \sum_{i<j,\ i\sim j} \left((x_i - x_j)^2 + (y_i - y_j)^2 \right),$$

where $i \sim j$ means that nodes i and j are connected by an edge. The objective J is precisely the sum of the squares of the lengths (in $\mathbb{R}^2$) of the drawn edges of the graph. We have to introduce some other constraints into our problem to get a sensible solution. First of all, the objective J is not affected if we shift all the coordinates by some fixed amount (since J only depends on differences of the x- and y-coordinates). So we can assume that

$$\sum_{i=1}^n x_i = 0, \qquad \sum_{i=1}^n y_i = 0,$$

i.e., the sum (or mean value) of the x- and y-coordinates is zero. These two equations 'center' our drawn graph. Another problem is that we can minimize J by putting all the nodes at $x_i = 0$, $y_i = 0$, which results in $J = 0$. To force the nodes to spread out, we impose the constraints

$$\sum_{i=1}^n x_i^2 = 1, \qquad \sum_{i=1}^n y_i^2 = 1, \qquad \sum_{i=1}^n x_i y_i = 0.$$

The first two say that the variance of the x- and y- coordinates is one; the last says that the x- and y- coordinates are uncorrelated. (You don't have to know what variance or uncorrelated mean; these are just names for the equations given above.) The three equations above enforce 'spreading' of the drawn graph. Even with these constraints, the coordinates that minimize J are not unique. For example, if x and y are any set of coordinates, and $Q \in \mathbb{R}^{2\times 2}$ is any orthogonal matrix, then the coordinates given by

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} = Q \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \qquad i = 1, \ldots, n$$

satisfy the centering and spreading constraints, and have the same value of J. This means that if you have a proposed set of coordinates for the nodes, then by rotating or reflecting them, you get another set of coordinates that is just as good, according to our objective. We'll just live with this ambiguity. Here's the question:

a) Explain how to solve this problem, *i.e.*, how to find x and y that minimize J subject to the centering and spreading constraints, given the graph topology. You can use any method or ideas we've encountered in the course. Be clear as to whether your approach solves the problem exactly (*i.e.*, finds a set of coordinates with J as small as it can possibly be), or whether it's just a good heuristic (*i.e.*, a choice of coordinates that achieves a reasonably small value of J, but perhaps not the absolute best). In describing your method, you may not refer to any programming commands or operators; your description must be entirely in mathematical terms.

b) Implement your method, and carry it out for the graph given in `dg_data.json`. This JSON file contains the *node adjacency matrix* of the graph, denoted A, and defined as $A_{ij} = 1$ if nodes i and j are connected by an edge, and $A_{ij} = 0$ otherwise. (The graph is undirected, so A is symmetric. Also, we do not have self-loops, so $A_{ii} = 0$, for $i = 1, \ldots, n$.) Give the value of J achieved by your choice of x and y, and verify that your x and y satisfy the centering and spreading conditions, at least approximately. If your method is iterative, plot the value of J versus iteration. Draw the corresponding graph by plotting nodes as small circles and edges as lines. For comparison, the JSON file also contains the vectors `x_circ` and `y_circ`. These coordinates were obtained using a standard technique for drawing a graph, by placing the nodes in order on a circle. The radius of the circle has been chosen so that `x_circ` and `y_circ` satisfy the centering and spread constraints. Draw this graph on a separate plot.

Hint. You are welcome to use the results described below, without proving them. Let $A \in \mathbb{R}^{n \times n}$ be symmetric, with eigenvalue decomposition $A = \sum_{i=1}^{n} \lambda_i q_i q_i^\mathsf{T}$, with $\lambda_1 \geq \cdots \geq \lambda_n$, and $\{q_1, \ldots, q_n\}$ orthonormal. You know that a solution of the problem

$$\begin{array}{ll} \text{minimize} & x^\mathsf{T} A x \\ \text{subject to} & x^\mathsf{T} x = 1, \end{array}$$

where the variable is $x \in \mathbb{R}^n$, is $x = q_n$. The related maximization problem is

$$\begin{array}{ll} \text{maximize} & x^\mathsf{T} A x \\ \text{subject to} & x^\mathsf{T} x = 1 \end{array}$$

with variable $x \in \mathbb{R}^n$. A solution to this problem is $x = q_1$. Now consider the following generalization of the first problem:

$$\begin{array}{ll} \text{minimize} & \mathrm{trace}(X^\mathsf{T} A X) \\ \text{subject to} & X^\mathsf{T} X = I_k \end{array}$$

where the variable is $X \in \mathbb{R}^{n \times k}$, and I_k denotes the $k \times k$ identity matrix, and we assume $k \leq n$. The constraint means that the columns of X, say, $x_1, \ldots, x_k$, are orthonormal; the objective can be written in terms of the columns of X as $\mathrm{trace}(X^\mathsf{T} A X) = \sum_{i=1}^{k} x_i^\mathsf{T} A x_i$. A solution of this problem is $X = [q_{n-k+1} \cdots q_n]$. Note that when $k = 1$, this reduces to the first problem above. The related maximization problem is

$$\begin{array}{ll} \text{maximize} & \mathrm{trace}(X^\mathsf{T} A X) \\ \text{subject to} & X^\mathsf{T} X = I_k \end{array}$$

with variable $X \in \mathbb{R}^{n \times k}$. A solution of this problem is $X = [q_1 \cdots q_k]$.

8. **Two representations of an ellipsoid.** In the lectures, we saw two different ways of representing an ellipsoid, centered at 0, with non-zero volume. The first uses a quadratic form:

$$\mathcal{E}_1 = \left\{ x \,\middle|\, x^\mathsf{T} S x \leq 1 \right\},$$

with $S^\mathsf{T} = S > 0$. The second is as the image of a unit ball under a linear mapping:

$$\mathcal{E}_2 = \{ y \mid y = Ax, \|x\| \leq 1 \},$$

with $\det(A) \neq 0$.

a) Given S, explain how to find an A so that $\mathcal{E}_1 = \mathcal{E}_2$.

b) Given A, explain how to find an S so that $\mathcal{E}_1 = \mathcal{E}_2$.

c) What about uniqueness? Given S, explain how to find *all* A that yield $\mathcal{E}_1 = \mathcal{E}_2$. Given A, explain how to find *all* S that yield $\mathcal{E}_1 = \mathcal{E}_2$.

9. **Worst-case analysis of impact.** We consider a (time-invariant) linear dynamical system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_{\mathrm{init}},$$

with state $x(t) \in \mathbb{R}^n$, and input $u(t) \in \mathbb{R}^m$. We are interested in the state trajectory over the time interval $[0, T]$. In this problem the input u represents an *impact* on the system, so it has the form

$$u(t) = g\delta(t - T_{\mathrm{imp}}),$$

where $g \in \mathbb{R}^m$ is a vector that gives the direction and magnitude of the impact, and T_{imp} is the time of the impact. We assume that $0 \leq T_{\mathrm{imp}} \leq T_-$. ($T_{\mathrm{imp}} = T_-$ means that the impact occurs right at the end of the period of interest, and does affect $x(T)$.) We let $x_{\mathrm{nom}}(T)$ denote the state, at time $t = T$, of the linear system $\dot{x}_{\mathrm{nom}} = Ax_{\mathrm{nom}}$, $x_{\mathrm{nom}}(0) = x_{\mathrm{init}}$. The vector $x_{\mathrm{nom}}(T)$ is what the final state $x(T)$ of the system above would have been at time $t = T$, had the impact not occurred (*i.e.*, with $u = 0$). We are interested in the deviation D between $x(T)$ and $x_{\mathrm{nom}}(T)$, as measured by the norm:

$$D = \|x(T) - x_{\mathrm{nom}}(T)\|.$$

D measures how far the impact has shifted the state at time T. We would like to know how large D can be, over all possible impact directions and magnitudes no more than one (*i.e.*, $\|g\| \leq 1$), and over all possible impact times between 0 and T_-. In other words, we would like to know the maximum possible state deviation, at time T, due to an impact of magnitude no more than one. We'll call the choices of T_{imp} and g that maximize D the *worst-case impact time* and *worst-case impact vector*, respectively.

a) Explain how to find the worst-case impact time, and the worst-case impact vector, given the problem data A, B, x_{init}, and T. Your explanation should be as short and clear as possible. You can use any of the concepts we have encountered in the class. Your approach can include a simple numerical search (such as plotting a function of one variable to find its maximum), if needed. If either the worst-case impact time or the worst-case impact vector do not depend on some of the problem data (*i.e.*, A, B, x_{init}, and T) say so.

b) Get the data from `worst_case_impact_data.json`, which defines A, B, x_{init}, and T, and carry out the procedure described in part (a). Be sure to give us the worst-case impact time (with absolute precision of 0.01), the worst-case impact vector, and the corresponding value of D.