EE263 Homework 4

Fall 2023

3.530. Checking some range and nullspace conditions. Explain how to determine whether or not the following statements hold:

a) $\text{range}(A) = \text{range}(B)$.

b) $\text{range}(A) \perp \text{range}(B)$.

c) $\text{range}(A) \cap \text{range}(B) = \{0\}$.

d) $\text{range}(C) \subseteq \text{null}(B)$.

The matrices have dimensions $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{p\times m}$.

Your answer can involve standard matrix operations on the matrices above, such as addition, multiplication, transposition, concatenation (*i.e.*, building block matrices), and inversion, as well as a function $\text{rank}(X)$, that gives the rank of a matrix X, and $\det(X)$, which gives the determinant of a (square) matrix X.

For example, you might assert that (a) holds if and only if $\text{rank}([A\ B]) = m$. (This is not correct; it's just an example of what your answer might look like.)

You do not need to give a proof or long justification that your conditions are correct; a short one or two sentence explanation for each statement is fine. Points will be deducted from correct answers that are substantially longer than they need to be, or are confusing (to us).

4.600. Sensor integrity monitor. A suite of m sensors yields measurement $y \in \mathbb{R}^m$ of some vector of parameters $x \in \mathbb{R}^n$. When the system is operating normally (which we hope is almost always the case) we have $y = Ax$, where $m > n$. If the system or sensors fail, or become faulty, then we no longer have the relation $y = Ax$. We can exploit the redundancy in our measurements to help us identify whether such a fault has occured. We'll call a measurement y *consistent* if it has the form Ax for some $x \in \mathbb{R}^n$. If the system is operating normally then our measurement will, of course, be consistent. If the system becomes faulty, we hope that the resulting measurement y will become inconsistent, *i.e.*, not consistent. (If we are *really* unlucky, the system will fail in such a way that y is still consistent. Then we're out of luck.) A matrix $B \in \mathbb{R}^{k\times m}$ is called an *integrity monitor* if the following holds:

- $By = 0$ for any y which is consistent.
- $By \neq 0$ for any y which is inconsistent.

If we find such a matrix B, we can quickly check whether y is consistent; we can send an alarm if $By \neq 0$. Note that the first requirement says that every consistent y does not trip the alarm; the second requirement states that every inconsistent y does trip the alarm. Finally, the problem. Find an integrity monitor B for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ -2 & 1 & 3 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Your B should have the smallest k (*i.e.*, number of rows) as possible. As usual, you have to explain what you're doing, as well as giving us your explicit matrix B. You must also verify that the matrix you choose satisfies the requirements. *Hints:*

- You might find one or more of the Julia functions `nullspace` or `qr` useful. Then again, you might not; there are many ways to find such a B.
- When checking that your B works, don't expect to have By exactly zero for a consistent y; because of roundoff errors in computer arithmetic, it will be really, really small. That's OK.
- Be very careful typing in the matrix A. It's not just a random matrix.

4.830. True/false questions about linear algebra. Determine whether each of the following statements is true or false. In each case, give either a proof or a counterexample.

a) If Q has orthonormal columns, then $\|Q^\mathsf{T} w\| \leq \|w\|$ for all vectors w.

b) Suppose $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times q}$. If $\mathrm{null}(A) = \{0\}$ and $\mathrm{range}(A) \subset \mathrm{range}(B)$, then $p \leq q$.

c) If $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is invertible and $\mathrm{range}(V_1) = \mathrm{null}(A)$, then $\mathrm{null}(AV_2) = \{0\}$.

d) If $\mathrm{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = \mathrm{rank}(A) = \mathrm{rank}(B)$, then $\mathrm{range}(A) = \mathrm{range}(B)$.

e) Suppose $A \in \mathbb{R}^{m\times n}$. Then, $x \in \mathrm{null}(A^\mathsf{T})$ if and only if $x \notin \mathrm{range}(A)$.

f) Suppose A is invertible. Then, AB is not full rank if and only if B is not full rank.

g) If A is not full rank, then there is a nonzero vector x such that $Ax = 0$.

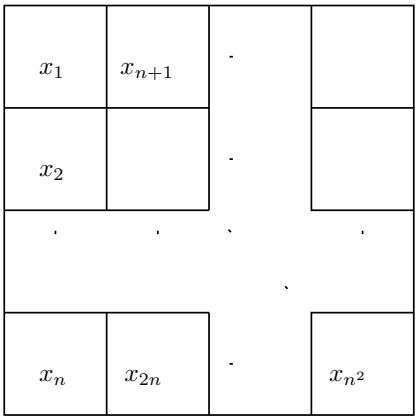
5.680. Least-squares residuals. Suppose A is skinny and full-rank. Let x_{ls} be the least-squares approximate solution of $Ax = y$, and let $y_{\mathrm{ls}} = Ax_{\mathrm{ls}}$. Show that the residual vector $r = y - y_{\mathrm{ls}}$ satisfies

$$\|r\|^2 = \|y\|^2 - \|y_{\mathrm{ls}}\|^2.$$

Also, give a brief geometric interpretation of this equality (just a couple of sentences, and maybe a conceptual drawing).

6.741. Image reconstruction from line integrals. In this problem we explore a simple version of a tomography problem. We consider a square region, which we divide into an $n \times n$ array

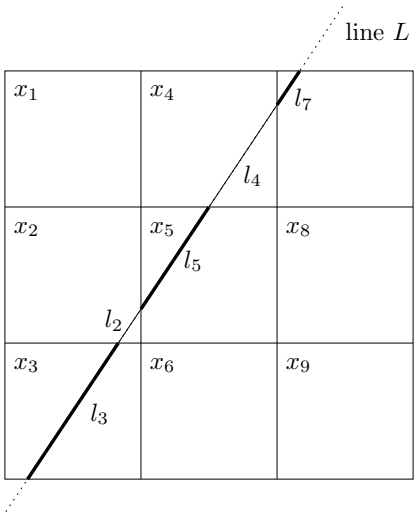
of square pixels, as shown below.



The pixels are indexed column first, by a single index i ranging from 1 to n^2, as shown above. We are interested in some physical property such as density (say) which varies over the region. To simplify things, we'll assume that the density is constant inside each pixel, and we denote by x_i the density in pixel i, $i = 1, \ldots, n^2$. Thus, $x \in \mathbb{R}^{n^2}$ is a vector that describes the density across the rectangular array of pixels. The problem is to estimate the vector of densities x, from a set of sensor measurements that we now describe. Each sensor measurement is a *line integral* of the density over a line L. In addition, each measurement is corrupted by a (small) noise term. In other words, the sensor measurement for line L is given by

$$\sum_{i=1}^{n^2} l_i x_i + v,$$

where l_i is the length of the intersection of line L with pixel i (or zero if they don't intersect), and v is a (small) measurement noise. This is illustrated below for a problem with $n = 3$. In this example, we have $l_1 = l_6 = l_8 = l_9 = 0$.



Now suppose we have N line integral measurements, associated with lines $L_1, \ldots, L_N$. From these measurements, we want to estimate the vector of densities x. The lines are characterized by the intersection lengths

$$l_{ij}, \quad i = 1, \ldots, n^2, \quad j = 1, \ldots, N,$$

where l_{ij} gives the length of the intersection of line L_j with pixel i. Then, the whole set of measurements forms a vector $y \in \mathbb{R}^N$ whose elements are given by

$$y_j = \sum_{i=1}^{n^2} l_{ij} x_i + v_j, \quad j = 1, \ldots, N.$$

And now the problem: you will reconstruct the pixel densities x from the line integral measurements y. The class webpage contains the file `tomo_data.json`, which contains the following variables:

- `N`, the number of measurements (N),
- `npixels`, the side length in pixels of the square region (n),
- `y`, a vector with the line integrals y_j, $j = 1, \ldots, N$,
- `line_pixel_lengths`, an $n^2 \times N$ matrix containing the intersection lengths l_{ij} of each pixel $i = 1, \ldots, n^2$ (ordered column-first as in the above diagram) and each line $j = 1, \ldots, N$,
- `lines_d`, a vector containing the displacement (distance from the center of the region in pixel lengths) d_j of each line $j = 1, \ldots, N$, and
- `lines_theta`, a vector containing the angles θ_j of each line $j = 1, \ldots, N$.

(You shouldn't need `lines_d` or `lines_theta`, but we're providing them to give you some idea of how the data was generated. Similarly, the file `tmeasure.jl` shows how we computed the measurements, but you don't need it or anything in it to solve the problem. The variable `line_pixel_lengths` was computed using the function in this file.)

Use this information to find x, and display it as an image (of n by n pixels). You'll know you have it right.

Julia hints:

- The `reshape` function might help with converting between vectors and matrices, for example, `A = reshape(v, m, n)` will convert a vector with $v = mn$ elements into an $m \times n$ matrix.
- To display a matrix `A` as a grayscale image, you can use: (or any method that works for you)

```
heatmap(A, yflip=true, aspect_ratio=:equal, color=:gist_gray,
        cbar=:none, framestyle=:none)
```

  You'll need to have loaded the JuliaPlots package with `using Plots` to access the `heatmap` function. (The `yflip` argument gets it to plot the origin in the top-left rather than the bottom-left.)

Note: While irrelevant to your solution, this is actually a simple version of *tomography*, best known for its application in medical imaging as the CAT scan. If an *x-ray* gets attenuated at rate x_i in pixel i (a little piece of a cross-section of your body), the j-th measurement is

$$z_j = \prod_{i=1}^{n^2} e^{-x_i l_{ij}},$$

with the l_{ij} as before. Now define $y_j = -\log z_j$, and we get

$$y_j = \sum_{i=1}^{n^2} x_i l_{ij}.$$

7.1060. Curve-smoothing. We are given a function $F : [0,1] \to \mathbb{R}$ (whose graph gives a curve in $\mathbb{R}^2$). Our goal is to find another function $G : [0,1] \to \mathbb{R}$, which is a *smoothed* version of F. We'll judge the smoothed version G of F in two ways:

- *Mean-square deviation from F,* defined as

$$D = \int_0^1 (F(t) - G(t))^2 \, dt.$$

- *Mean-square curvature,* defined as

$$C = \int_0^1 G''(t)^2 \, dt.$$

We want *both* D and C to be small, so we have a problem with two objectives. In general there will be a trade-off between the two objectives. At one extreme, we can choose $G = F$, which makes $D = 0$; at the other extreme, we can choose G to be an affine function (*i.e.*, to have $G''(t) = 0$ for all $t \in [0,1]$), in which case $C = 0$. The problem is to identify the optimal trade-off curve between C and D, and explain how to find smoothed functions G on the optimal trade-off curve. To reduce the problem to a finite-dimensional one, we will represent the functions F and G (approximately) by vectors $f,\ g \in \mathbb{R}^n$, where

$$f_i = F(i/n), \quad g_i = G(i/n).$$

You can assume that n is chosen large enough to represent the functions well. Using this representation we will use the following objectives, which approximate the ones defined for the functions above:

- *Mean-square deviation,* defined as

$$d = \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)^2.$$

- *Mean-square curvature,* defined as

$$c = \frac{1}{n-2} \sum_{i=2}^{n-1} \left(\frac{g_{i+1} - 2g_i + g_{i-1}}{1/n^2} \right)^2.$$

In our definition of c, note that

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{1/n^2}$$

gives a simple approximation of $G''(i/n)$. You will only work with this approximate version of the problem, *i.e.*, the vectors f and g and the objectives c and d.

a) Explain how to find g that minimizes $d + \mu c$, where $\mu \geq 0$ is a parameter that gives the relative weighting of sum-square curvature compared to sum-square deviation. Does your method always work? If there are some assumptions you need to make (say, on rank of some matrix, independence of some vectors, etc.), state them clearly. Explain how to obtain the two extreme cases: $\mu = 0$, which corresponds to minimizing d without regard for c, and also the solution obtained as $\mu \to \infty$ (*i.e.*, as we put more and more weight on minimizing curvature).

b) Get the file `curve_smoothing.json` from the course web site. This file defines a specific vector f that you will use. Find and plot the optimal trade-off curve between d and c. Be sure to identify any critical points (such as, for example, any intersection of the curve with an axis). Plot the optimal g for the two extreme cases $\mu = 0$ and $\mu \to \infty$, and for three values of μ in between (chosen to show the trade-off nicely). On your plots of g, be sure to include also a plot of f, say with dotted line type, for reference.