3.670. Zeroing out the board. Bobbie and Reza are playing a game. This game is played on a 6 \times 6 board as follows. First, Bobbie fills the board with 36 arbitrary real numbers and then Reza performs a sequence of actions:

At each step, Reza is allowed to choose one cell, and an arbitrary real number $x$. Then he can add $x$ to the selected cell and subtract $x$ from all adjacent cells. (Some cells have four adjacent cells, some have three, and some have two.) Reza’s goal to perform a sequence of allowed actions to derive a table which consists of 36 zeros. If Reza can derive the table consisting of zeros, he wins the game, otherwise Bobbie is the winner.

a) If Bobbie writes 1 in a corner cell (and 0 elsewhere), can Reza win the game? If you believe the answer is positive, you should specify the sequence of actions Reza should take. If your answer is negative, you should prove that there is no possible sequence of actions that Reza can take to zero out the table.

b) Can Bobbie fill in the table so that Reza has no possible way of winning the game? If your answer is positive, you should prove that there exists an initial table that Reza cannot turn into zero. If your answer is negative, you should prove that Reza can turn any initial table into zero with a sequence of allowed actions.

c) Solve part b for a 9 \times 9 table.

4.570. Orthogonal matrices.

a) Show that if $U$ and $V$ are orthogonal, then so is $UV$.

b) Show that if $U$ is orthogonal, then so is $U^{-1}$.

c) Suppose that $U \in \mathbb{R}^{2 \times 2}$ is orthogonal. Show that $U$ is either a rotation or a reflection. Make clear how you decide whether a given orthogonal $U$ is a rotation or reflection.

4.610. Householder reflections. A Householder matrix is defined as

$$Q = I - 2uu^T,$$

where $u \in \mathbb{R}^n$ is normalized, that is, $u^Tu = 1$.

a) Show that $Q$ is orthogonal.

b) Show that $Qu = -u$. Show that $Qv = v$, for any $v$ such that $u^Tv = 0$. Thus, multiplication by $Q$ gives reflection through the plane with normal vector $u$.

c) Given a vector $x \in \mathbb{R}^n$, find a unit-length vector $u$ for which $Qx$ lies on the line through $e_1$. *Hint:* Try a $u$ of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate $\alpha$ and show that such a $u$ works . . .) Compute such a $u$ for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to $x$ to find $Qx$.

*Note:* Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.
5.680. **Least-squares residuals.** Suppose $A$ is skinny and full-rank. Let $x_{ls}$ be the least-squares approximate solution of $Ax = y$, and let $y_{ls} = Ax_{ls}$. Show that the residual vector $r = y - y_{ls}$ satisfies
\[ ||r||^2 = ||y||^2 - ||y_{ls}||^2. \]

Also, give a brief geometric interpretation of this equality (just a couple of sentences, and maybe a conceptual drawing).

6.720. **Approximate inductance formula.** The figure below shows a planar spiral inductor, implemented in CMOS, for use in RF circuits.

![Planar Spiral Inductor Diagram](image)

The inductor is characterized by four key parameters:
- $n$, the number of turns (which is a multiple of $1/4$, but that needn’t concern us)
- $w$, the width of the wire
- $d$, the inner diameter
- $D$, the outer diameter

The inductance $L$ of such an inductor is a complicated function of the parameters $n$, $w$, $d$, and $D$. The inductance $L$ can be found by solving Maxwell’s equations, which takes considerable computer time, or by fabricating the inductor and measuring the inductance. In this problem you will develop a simple approximate inductance model of the form
\[ \hat{L} = \alpha n^{\beta_1} w^{\beta_2} d^{\beta_3} D^{\beta_4}, \]
where $\alpha, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}$ are constants that characterize the approximate model. (since $L$ is positive, we have $\alpha > 0$, but the constants $\beta_2, \ldots, \beta_4$ can be negative.) This simple approximate model, if accurate enough, can be used for design of planar spiral inductors.

The file `inductor_data.json` on the course web site contains data for 50 inductors. (The data is real, not that it would affect how you solve the problem . . .) For inductor $i$, we give parameters $n_i$, $w_i$, $d_i$, and $D_i$ (all in $\mu$m), and also, the inductance $L_i$ (in nH) obtained from
measurements. (The data are organized as vectors of length 50. Thus, for example, $w_{13}$ gives the wire width of inductor 13.) Your task, i.e., the problem, is to find $\alpha, \beta_1, \ldots, \beta_4$ so that

$$\hat{L}_i = \alpha n_i^{\beta_1} w_i^{\beta_2} d_i^{\beta_3} D_i^{\beta_4} \approx L_i \quad \text{for } i = 1, \ldots, 50.$$ 

Your solution must include a clear description of how you found your parameters, as well as their actual numerical values. Note that we have not specified the criterion that you use to judge the approximate model (i.e., the fit between $\hat{L}_i$ and $L_i$); we leave that to your engineering judgment. But be sure to tell us what criterion you use. We define the percentage error between $\hat{L}_i$ and $L_i$ as

$$e_i = 100|\hat{L}_i - L_i|/L_i.$$ 

Find the average percentage error for your model, i.e., $(e_1 + \cdots + e_{50})/50$. (We are only asking you to find the average percentage error for your model; we do not require that your model minimize the average percentage error.) Hint: you might find it easier to work with $\log L$. 

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