EE263 Homework 3

Fall 2023

3.250. Color perception. Human color perception is based on the responses of three different types of color light receptors, called *cones*. The three types of cones have different spectral-response characteristics, and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' responses as follows:

$$L_{\text{cone}} = \sum_{i=1}^{20} l_i p_i, \qquad M_{\text{cone}} = \sum_{i=1}^{20} m_i p_i, \qquad S_{\text{cone}} = \sum_{i=1}^{20} s_i p_i,$$

where p_i is the incident power in the ith wavelength band, and l_i, m_i and s_i are nonnegative constants that describe the spectral responses of the different cones. The perceived color is a complex function of the three cone responses, *i.e.*, the vector $(L_{\text{cone}}, M_{\text{cone}}, S_{\text{cone}})$, with different cone response vectors perceived as different colors. (Actual color perception is a bit more complicated than this, but the basic idea is right.)

a) *Metamers.* When are two light spectra, p and $\tilde{p}$, visually indistinguishable? (Visually identical lights with different spectral power compositions are called *metamers*.)

b) *Visual color matching.* In a color matching problem, an observer is shown a test light, and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked the find a spectrum of the form

$$p_{\text{match}} = a_1 u + a_2 v + a_3 w,$$

where u, v, w are the spectra of the primary lights, and a_i are the intensities to be found, that is visually indistinguishable from a given test light spectrum p_{test}. Can this always be done? Discuss briefly.

c) *Visual matching with phosphors.* A computer monitor has three phosphors, R, G, and B. It is desired to adjust the phosphor intensities to create a color that looks like a reference test light. Find weights that achieve the match or explain why no such weights exist. The data for this problem is in `color_perception_data.json`, which contains the vectors `wavelength`, `B_phosphor`, `G_phosphor`, `R_phosphor`, `L_coefficients`, `M_coefficients`, `S_coefficients`, and `test_light`.

d) *Effects of illumination.* An object's surface can be characterized by its reflectance (*i.e.*, the fraction of light it reflects) for each band of wavelengths. If the object is illuminated with a light spectrum characterized by I_i, and the reflectance of the object is r_i (which is between 0 and 1), then the reflected light spectrum is given by $I_i r_i$, where $i = 1, \ldots, 20$ denotes the wavelength band. Now consider two objects illuminated (at different times) by two different light sources, say an incandescent bulb and sunlight. Sally argues that if the two objects look identical when illuminated by a tungsten bulb, then they will look identical when illuminated by sunlight. Beth disagrees: she says that two objects can appear identical when illuminated by a tungsten bulb, but look different when lit by sunlight. Who is right? If Sally is right, explain why. If Beth is right give an example of two objects that appear identical under one light source and different under another. You can use the vectors `sunlight` and `tungsten` defined in the data file as the light sources.

Remark. Spectra, intensities, and reflectances are all nonnegative quantities, which the material of EE263 doesn't address. So just ignore this while doing this problem. These issues can be handled using the material of EE364a, however.

3.300. Orthogonal complement of a subspace. If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ we define $\mathcal{V}^\perp$ as the set of vectors orthogonal to every element in $\mathcal{V}$, *i.e.*,

$$\mathcal{V}^\perp = \{\ x \mid \langle x, y \rangle = 0,\ \forall y \in \mathcal{V}\ \}.$$

a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

b) Suppose $\mathcal{V}$ is described as the span of some vectors $v_1, v_2, \ldots, v_r$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ in terms of the matrix $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} \in \mathbb{R}^{n \times r}$ using common terms (range, nullspace, transpose, etc.)

c) Show that every $x \in \mathbb{R}^n$ can be expressed uniquely as $x = v + v^\perp$ where $v \in \mathcal{V}$, $v^\perp \in \mathcal{V}^\perp$. *Hint:* let v be the projection of x on $\mathcal{V}$.

d) Show that $\dim \mathcal{V}^\perp + \dim \mathcal{V} = n$.

e) Show that $\mathcal{V} \subseteq \mathcal{U}$ implies $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$.

3.430. Single sensor failure detection and identification. We have $y = Ax$, where $A \in \mathbb{R}^{m \times n}$ is known, and $x \in \mathbb{R}^n$ is to be found. Unfortunately, up to one sensor may have failed (but you don't know which one has failed, or even whether any has failed). You are given $\tilde{y}$ and not y, where $\tilde{y}$ is the same as y in all entries except, possibly, one (say, the kth entry). If all sensors are operating correctly, we have $y = \tilde{y}$. If the kth sensor fails, we have $\tilde{y}_i = y_i$ for all $i \neq k$.

The file `one_bad_sensor.json`, available on the course web site, defines A and $\tilde{y}$ (as `A` and `ytilde`). Determine which sensor has failed (or if no sensors have failed). You must explain your method, and submit your code.

For this exercise, you can use the matlab code `rank([F g])==rank(F)` to check if $g \in \text{range}(F)$. (We will see later a much better way to check if $g \in \text{range}(F)$.)

3.680. Coin collector robot. Consider a robot with unit mass which can move in a frictionless two dimensional plane. The robot has a constant unit speed in the y direction (towards north), and it is designed such that we can only apply force in the x direction. We will apply a force at time t given by f_j for $2j - 2 \leq t < 2j$ where $j = 1, \ldots, n$, so that the applied force is constant over time intervals of length 2. The robot is at the origin at time $t = 0$ with zero velocity in the x direction.

There are $2n$ coins in the plane and the goal is to design a sequence of input forces for the robot to collect the maximum possible number of coins. The robot is designed such that it can collect the ith coin only if it exactly passes through the location of the coin (x_i, y_i). In this problem, we assume that $y_i = i$.

a) Find the coordinates of the robot at time t, where t is a positive integer. Your answer should be a function of t and the vector of input forces $f \in \mathbf{R}^n$.

b) Given a sequence of k coins $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$, describe a method to find whether the robot can collect them.

c) For the data provided in `robot_coin_collector.json`, show that the robot cannot collect all the coins.

d) Suppose that there is an arrangement of the coins such that it is not possible for the robot to collect all the coins. Suggest a way to check if the robot can collect all but one of the coins.

e) Run your method on data in `robot_coin_collector.json` and report which coin cannot be collected. Report the input that results in collecting $2n-1$ coins. Plot the location of the coins and the location of the robot at integer times.

4.630. Groups of equivalent statements. In the list below there are 11 statements about two square matrices A and B in $\mathbb{R}^{n\times n}$.

a) $\text{range}(B) \subseteq \text{range}(A)$.

b) there exists a matrix $Y \in \mathbb{R}^{n\times n}$ such that $B = YA$.

c) $AB = 0$.

d) $BA = 0$.

e) $\text{rank}(\begin{bmatrix} A & B \end{bmatrix}) = \text{rank}(A)$.

f) $\text{range}(A) \perp \text{null}(B^\mathsf{T})$.

g) $\text{rank}(\begin{bmatrix} A \\ B \end{bmatrix}) = \text{rank}(A)$.

h) $\text{range}(A) \subseteq \text{null}(B)$.

i) there exists a matrix $Z \in \mathbb{R}^{n\times n}$ such that $B = AZ$.

j) $\text{rank}(\begin{bmatrix} A & B \end{bmatrix}) = \text{rank}(B)$.

k) $\text{null}(A) \subseteq \text{null}(B)$.

Your job is to collect them into (the largest possible) groups of equivalent statements. Two statements are equivalent if each one implies the other. For example, the statement 'A is onto' is equivalent to '$\text{null}(A) = \{0\}$' (when A is square, which we assume here), because every square matrix that is onto has zero nullspace, and vice versa. Two statements are not equivalent if there exist (real) square matrices A and B for which one holds, but the other does not. A group of statements is equivalent if any pair of statements in the group is equivalent.

We want *just* your answer, which will consist of lists of mutually equivalent statements; we do not need any justification.

Put your answer in the following specific form. List each group of equivalent statements on a line, in (alphabetic) order. Each new line should start with the first letter not listed above. For example, you might give your answer as

a, c, d, h

b, i

e

f, g, j, k.

This means you believe that statements a, c, d, and h are equivalent; statements b and i are equivalent; and statements f, g, j, and k are equivalent. You also believe that the first group of statements is not equivalent to the second, or the third, and so on.