Homework 2

EE 263 Stanford University Summer 2018

Due: July 11, 2018

1. Halfspace. Suppose $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, *i.e.*:

$$\{x \mid \|x - a\| \leq \|x - b\| \ \} = \{ \ x \mid c^\mathsf{T} x \leq d\}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

2. Some properties of the product of two matrices. For each of the following statements, either show that it is true, or give a (specific) counterexample.

- If AB is full rank then A and B are full rank.
- If A and B are full rank then AB is full rank.
- If A and B have zero nullspace, then so does AB.
- If A and B are onto, then so is AB.

You can assume that $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. Some of the false statements above become true under certain assumptions on the dimensions of A and B. As a trivial example, all of the statements above are true when A and B are scalars, *i.e.*, $n = m = p = 1$. For each of the statements above, find conditions on n, m, and p that make them true. Try to find the most general conditions you can. You can give your conditions as inequalities involving n, m, and p, or you can use more informal language such as "A and B are both skinny."

3. Geometric analysis of navigation. Let $(x, y) \in \mathbb{R}^2$ be the unknown coordinates of a point in the plane, let $(p_i, q_i) \in \mathbb{R}^2$ be the known coordinates of a beacon for $i = 1, \ldots, n$, and let ρ_i be the measured distance between (x, y) and the ith beacon. The linearized navigation equations near a point $(x_0, y_0) \in \mathbb{R}^2$ are

$$\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix},$$

where we define the matrix $A \in \mathbb{R}^{n \times 2}$ such that

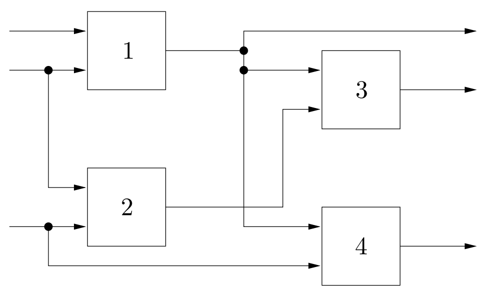
$$A_{i1} = \frac{x_0 - p_i}{\|(x_0, y_0) - (p_i, q_i)\|} \qquad \text{and} \qquad A_{i2} = \frac{y_0 - q_i}{\|(x_0, y_0) - (p_i, q_i)\|}.$$

Find the conditions under which A has full rank. Describe the conditions geometrically (*i.e.*, in terms of the relative positions of the unknown coordinates and the beacons).

4. **Digital circuit gate sizing.** A digital circuit consists of a set of n (logic) gates, interconnected by wires. Each gate has one or more inputs (typically between one and four), and one output, which is connected via the wires to other gate inputs and possibly to some external circuitry. When the output of gate i is connected to an input of gate j, we say that gate i *drives* gate j, or that gate j is in the *fan-out* of gate i. We describe the topology of the circuit by the *fan-out list* for each gate, which tells us which other gates the output of a gate connects to. We denote the fan-out list of gate i as $\text{FO}(i) \subseteq \{1, \ldots, n\}$. We can have $\text{FO}(i) = \emptyset$, which means that the output of gate i does not connect to the inputs of any of the gates $1, \ldots, n$ (presumably the output of gate i connects to some external circuitry). It's common to order the gates in such a way that each gate only drives gates with higher indices, *i.e.*, we have $\text{FO}(i) \subseteq \{i+1, \ldots, n\}$. We'll assume that's the case here. (This means that the gate interconnections form a directed acyclic graph.)

To illustrate the notation, a simple digital circuit with $n = 4$ gates, each with 2 inputs, is shown below. For this circuit we have

$$\text{FO}(1) = \{3, 4\}, \quad \text{FO}(2) = \{3\}, \quad \text{FO}(3) = \emptyset, \quad \text{FO}(4) = \emptyset.$$



The 3 input signals arriving from the left are called *primary inputs*, and the 3 output signals emerging from the right are called *primary outputs* of the circuit. (You don't need to know this, however, to solve this problem.)

Each gate has a (real) *scale factor* or *size* x_i. These scale factors are the design variables in the gate sizing problem. They must satisfy $1 \leq x_i \leq x^{\max}$, where $x^{\max}$ is a given maximum allowed gate scale factor (typically on the order of 100). The total area of the circuit has the form

$$A = \sum_{i=1}^{n} a_i x_i,$$

where a_i are positive constants.

Each gate has an *input capacitance* C_i^{in}, which depends on the scale factor x_i as

$$C_i^{\text{in}} = \alpha_i x_i,$$

where α_i are positive constants.

Each gate has a *delay* d_i, which is given by

$$d_i = \beta_i + \gamma_i C_i^{\text{load}} / x_i,$$

where β_i and γ_i are positive constants, and C_i^{load} is the *load capacitance* of gate i. Note that the gate delay d_i is always larger than β_i, which can be intepreted as the minimum possible delay of gate i, achieved only in the limit as the gate scale factor becomes large.

The load capacitance of gate i is given by

$$C_i^{\text{load}} = C_i^{\text{ext}} + \sum_{j \in \text{FO}(i)} C_j^{\text{in}},$$

where C_i^{ext} is a positive constant that accounts for the capacitance of the interconnect wires and external circuitry.

We will follow a simple design method, which assigns an equal delay T to all gates in the circuit, *i.e.*, we have $d_i = T$, where $T > 0$ is given. For a given value of T, there may or may not exist a feasible design (*i.e.*, a choice of the x_i, with $1 \leq x_i \leq x^{\max}$) that yields $d_i = T$ for $i = 1, \ldots, n$. We can assume, of course, that $T > \max_i \beta_i$, *i.e.*, T is larger than the largest minimum delay of the gates.

Finally, we get to the problem.

a) Explain how to find a design $x^\star \in \mathbb{R}^n$ that minimizes T, subject to a given area constraint $A \leq A^{\max}$. You can assume the fanout lists, and all constants in the problem description are known; your job is to find the scale factors x_i. Be sure to explain how you determine if the design problem is feasible, *i.e.*, whether or not there is an x that gives $d_i = T$, with $1 \leq x_i \leq x^{\max}$, and $A \leq A^{\max}$.

   Your method can involve any of the methods or concepts we have seen so far in the course. It can also involve a simple search procedure, *e.g.*, trying (many) different values of T over a range.

   Note: this problem concerns the general case, and not the simple example shown above.

b) Carry out your method on the particular circuit with data given in the file `gate_sizing_data.json`

   The fan-out lists are given as an $n \times n$ matrix `F`, with i, j entry one if $j \in \text{FO}(i)$, and zero otherwise. In other words, the ith row of `F` gives the fanout of gate i. The jth entry in the ith row is 1 if gate j is in the fan-out of gate i, and 0 otherwise.

Comment. You do not need to know anything about digital circuits; *everything* you need to know is stated above.

5. Projection matrices. A matrix $P \in \mathbb{R}^{n \times n}$ is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$.

a) Show that if P is a projection matrix then so is $I - P$.

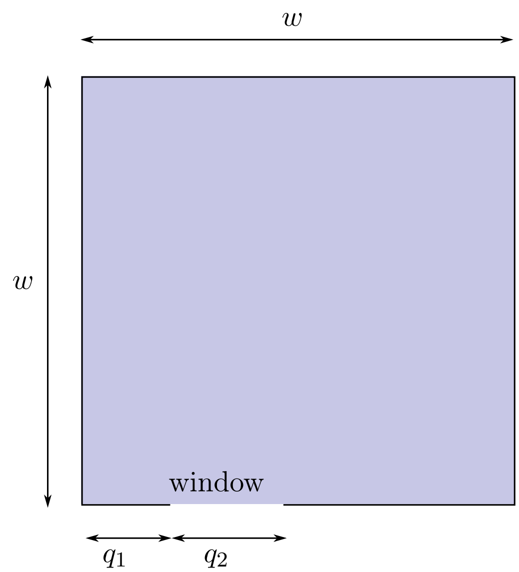
b) Suppose that the columns of $U \in \mathbb{R}^{n \times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

c) Suppose $A \in \mathbb{R}^{n \times k}$ is full rank, with $k \leq n$. Show that $A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is a projection matrix.

d) If $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the point y in S closest to x is called the *projection of x on S*. Show that if P is a projection matrix, then $y = Px$ is the projection of x on range(P). (Which is why such matrices are called projection matrices ...)

6. Light reflection from diffuse surfaces. We consider a simple model for the reflection of light from a surface. The idea of the model is that the amount of energy flux leaving a surface is the sum of the amount produced by the surface (if it is a light emitter) plus the amount reflected from the surface. To simplify matters, we'll assume that the surfaces are *Lambertian*, that is, a single ray of light incident on the surface is reflected equally in all directions, so no matter where we view the surface from it will look equally bright. This is a good model for surfaces that are not shiny.

You are an architect, charged with determining how the light in a room with one window will look like. We'll work in a two-dimensional world (which simplifies much of the drudgery of computing meshes). A horizontal slice through the room is below.



The room has four walls, and the south wall has a window in it, of width q_2, placed at a distance q_1 from the west wall. We divide each wall up into segments called elements. The total light flux from element i, called the *radiosity* of i, will be denoted by x_i. This light is emitted in all directions, and the light flux from element j that hits element i is given by $F_{ij}x_j$. The number F_{ij} is called the *form factor*, and only depends on the geometry. It turns out that there is a simple formula for the form factor, given by the *strings rule.* (You don't need to be able to prove this.) Specifically,

$$F_{ij} = \frac{AD + BC - AC - BD}{2l_i}$$

where l_i is the length of element i, and AD denotes the length of the line from point A to point D, as shown in the figure below. That is, F_{ij} is the sum of the lengths of the 'crossed strings' minus the sum of the lengths of the 'uncrossed strings', divided by $2l_i$. Since all light must go somewhere, we have

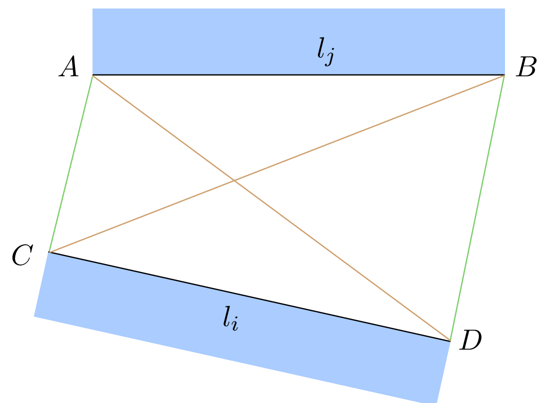
$$\sum_i F_{ij} = 1$$

where F_{ii} is defined to be zero. The light reflected from element i is then

$$\rho_i \sum_j F_{ij} x_j$$

where ρ_i is the reflectivity coefficient of the element i. The window does not reflect light, but it emits light, with a flux of 1 unit per element. Light enters through the window, bounces around the room, and the walls are therefore illuminated nonuniformly depending on position.

We will have $w = 100$, $q_1 = 20$, $q_2 = 20$, and $\rho_i = 0.8$ for the room walls. We will use elements of length 1, so in total we will divide the walls (plus window) of the room up into 400 elements.



a) Let x be the vector of radiosities of each of the elements. What is the equation that determines the radiosity x?

b) Find x, and plot the radiosity of each of the walls as a function of position along the wall.

c) Which point on the walls is the brightest (excluding the window)? What is the radiosity there?

7. True/false questions about linear algebra. Determine whether each of the following statements is true or false. In each case, give either a proof or a counterexample.

a) If Q has orthonormal columns, then $\|Q^\mathsf{T} w\| \leq \|w\|$ for all vectors w.

b) Suppose $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{m \times q}$. If $\text{null}(A) = \{0\}$ and $\text{range}(A) \subset \text{range}(B)$, then $p \leq q$.

c) If $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is invertible and $\text{range}(V_1) = \text{null}(A)$, then $\text{null}(AV_2) = \{0\}$.

d) If $\text{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = \text{rank}(A) = \text{rank}(B)$, then $\text{range}(A) = \text{range}(B)$.

e) Suppose $A \in \mathbb{R}^{m \times n}$. Then, $x \in \text{null}(A^\mathsf{T})$ if and only if $x \notin \text{range}(A)$.

f) Suppose A is invertible. Then, AB is not full rank if and only if B is not full rank.

g) If A is not full rank, then there is a nonzero vector x such that $Ax = 0$.