Homework 2

EE263 Stanford University, Fall 2017

Due: Wednesday 10/11/17 11:59pm

1. **Quadratic extrapolation of a time series.** We are given a series z up to time t. Using a quadratic model, we want to extrapolate, or predict, $z(t+1)$ based on the three previous elements of the series, $z(t)$, $z(t-1)$, and $z(t-2)$. We'll denote the predicted value of $z(t+1)$ by $\hat{z}(t+1)$. More precisely, you will find $\hat{z}(t+1)$ as follows.

   a) Find the quadratic function $f(\tau) = a_2\tau^2 + a_1\tau + a_0$ which satisfies $f(t) = z(t)$, $f(t-1) = z(t-1)$, and $f(t-2) = z(t-2)$. Then the extrapolated value is given by $\hat{z}(t+1) = f(t+1)$. Show that

   $$\hat{z}(t+1) = c \begin{bmatrix} z(t) \\ z(t-1) \\ z(t-2) \end{bmatrix},$$

   where $c \in \mathbb{R}^{1\times 3}$, and does not depend on t. In other words, the quadratic extrapolator is a linear function. Find c explicitly.

   b) Use the following matlab code to generate a time series z:

   ```
   t = 1:1000;
   z = 5*sin(t/10 + 2) + 0.1*sin(t) + 0.1*sin(2*t - 5);
   ```

   Use the quadratic extrapolation method from part (a) to find $\hat{z}(t)$ for $t = 4, \ldots, 1000$. Find the relative root-mean-square (RMS) error, which is given by

   $$\left(\frac{(1/997)\sum_{j=4}^{1000} (\hat{z}(j) - z(j))^2}{(1/997)\sum_{j=4}^{1000} z(j)^2} \right)^{1/2}.$$

2. **Color perception.** Human color perception is based on the responses of three different types of color light receptors, called *cones*. The three types of cones have different spectral-response characteristics, and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' responses as follows:

   $$L_{\text{cone}} = \sum_{i=1}^{20} l_i p_i, \qquad M_{\text{cone}} = \sum_{i=1}^{20} m_i p_i, \qquad S_{\text{cone}} = \sum_{i=1}^{20} s_i p_i,$$

   where p_i is the incident power in the ith wavelength band, and l_i, m_i and s_i are nonnegative constants that describe the spectral responses of the different cones. The perceived color is a complex function of the three cone responses, *i.e.*, the vector $(L_{\text{cone}}, M_{\text{cone}}, S_{\text{cone}})$, with different cone response vectors perceived as different colors. (Actual color perception is a bit more complicated than this, but the basic idea is right.)

a) *Metamers.* When are two light spectra, p and $\tilde{p}$, visually indistinguishable? (Visually identical lights with different spectral power compositions are called *metamers*.)

b) *Visual color matching.* In a color matching problem, an observer is shown a test light, and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked the find a spectrum of the form

$$p_{\text{match}} = a_1 u + a_2 v + a_3 w,$$

where u, v, w are the spectra of the primary lights, and a_i are the intensities to be found, that is visually indistinguishable from a given test light spectrum p_{test}. Can this always be done? Discuss briefly.

c) *Visual matching with phosphors.* A computer monitor has three phosphors, R, G, and B. It is desired to adjust the phosphor intensities to create a color that looks like a reference test light. Find weights that achieve the match or explain why no such weights exist. The data for this problem is in `color_perception_data.m`, which contains the vectors `wavelength`, `B_phosphor`, `G_phosphor`, `R_phosphor`, `L_coefficients`, `M_coefficients`, `S_coefficients`, and `test_light`.

d) *Effects of illumination.* An object's surface can be characterized by its reflectance (*i.e.*, the fraction of light it reflects) for each band of wavelengths. If the object is illuminated with a light spectrum characterized by I_i, and the reflectance of the object is r_i (which is between 0 and 1), then the reflected light spectrum is given by $I_i r_i$, where $i = 1, \ldots, 20$ denotes the wavelength band. Now consider two objects illuminated (at different times) by two different light sources, say an incandescent bulb and sunlight. Sally argues that if the two objects look identical when illuminated by a tungsten bulb, then they will look identical when illuminated by sunlight. Beth disagrees: she says that two objects can appear identical when illuminated by a tungsten bulb, but look different when lit by sunlight. Who is right? If Sally is right, explain why. If Beth is right give an example of two objects that appear identical under one light source and different under another. You can use the vectors `sunlight` and `tungsten` defined in the data file as the light sources.

Remark. Spectra, intensities, and reflectances are all nonnegative quantities, which the material of EE263 doesn't address. So just ignore this while doing this problem. These issues can be handled using the material of EE364a, however.

3. Gambler's ruin. Consider a gambling situation involving two players A and B. An example is roulette where, say, player A is a *guest* and player B is the *house*. During any one play of the game there is a probability p, $0 < p < 1$, that player A wins a chip (or coin) from player B, and a probability $q = 1 - p$ that Player B wins a chip from player A. The players begin with initial holdings of a and b chips, respectively. A player wins overall if she obtains all the chips.

a) Find the probability that player A wins.
Hint. This might sound like a problem for a probability and statistics course, but we want you to approach this problem from a linear dynamical systems point of view. Consider the general situation where A has k chips. Denote the probability under these circumstances that player A eventually wins by $u(k)$. Assume $u(k)$ is the state of the system you are analyzing. Can you write a difference equation that describes the dynamics of $u(k)$? To solve your difference equation you can assume the solution has the general form $u(k) = \lambda^k$ (we will see why later in the class). You will also need to come up with two initial conditions to uniquely solve your difference equation. Think of $u(k)$ when player A has no chips, or has $a + b$ chips.

b) As a specific example, suppose you play a roulette wheel that has 37 divisions: 18 are red, 18 are black and one is green. If you bet on either red or black, you win a sum equal to your bet if the outcome is a division of that color (You cannot bet on green). Otherwise you loose your bet. If the bank has 1000 chips and you have 100 chips, what is the chance that you can *break the bank,* betting only one chip on red or black each spin of the wheel?

4. **Identifying a point on the unit sphere from spherical distances.** In this problem we consider the *unit sphere* in $\mathbb{R}^n$, which is defined as the set of vectors with norm one: $S^n = \{x \in \mathbb{R}^n \mid normx = 1\}$. We define the *spherical distance* between two vectors on the unit sphere as the distance between them, measured along the sphere, *i.e.*, as the angle between the vectors, measured in radians: If $x,\ y \in S^n$, the spherical distance between them is

$$\mathrm{sphdist}(x, y) = \angle(x, y),$$

where we take the angle as lying between 0 and π. (Thus, the maximum distance between two points in S^n is π, which occurs only when the two points x, y are *antipodal*, which means $x = -y$.) Now suppose $p_1, \ldots, p_k \in S^n$ are the (known) positions of some beacons on the unit sphere, and let $x \in S^n$ be an unknown point on the unit sphere. We have exact measurements of the (spherical) distances between each beacon and the unknown point x, *i.e.*, we are given the numbers

$$\rho_i = \mathrm{sphdist}(x, p_i), \quad i = 1, \ldots, k.$$

We would like to determine, without any ambiguity, the exact position of x, based on this information. Find the conditions on $p_1, \ldots, p_k$ under which we can unambiguously determine x, for any $x \in S^n$, given the distances ρ_i. You can give your solution algebraically, using any of the concepts used in class (*e.g.*, nullspace, range, rank), or you can give a geometric condition (involving the vectors p_i). You must justify your answer.

5. **Some true/false questions.** Determine if the following statements are true or false. No justification or discussion is needed for your answers. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that

the dimensions are such that all expressions make sense. For example, the statement "$A+B = B+A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.)

a) If all coefficients (*i.e.*, entries) of the matrix A are positive, then A is full rank.

b) If A and B are onto, then $A+B$ must be onto.

c) If A and B are onto, then so is the matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

d) If A and B are onto, then so is the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$.

e) If the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is onto, then so are the matrices A and B.

f) If A is full rank and skinny, then so is the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$.

6. Temperatures in a multi-core processor. We are concerned with the temperature of a processor at two critical locations. These temperatures, denoted $T = (T_1, T_2)$ (in degrees C), are affine functions of the power dissipated by three processor cores, denoted $P = (P_1, P_2, P_3)$ (in W). We make 4 measurements. In the first, all cores are idling, and dissipate 10W. In the next three measurements, one of the processors is set to full power, 100W, and the other two are idling. In each experiment we measure and note the temperatures at the two critical locations.

P_1	P_2	P_3	T_1	T_2
10W	10W	10W	27°	29°
100W	10W	10W	45°	37°
10W	100W	10W	41°	49°
10W	10W	100W	35°	55°

Suppose we operate all cores at the same power, p. How large can we make p, without T_1 or T_2 exceeding 70°?

You must fully explain your reasoning and method, in addition to providing the numerical solution.

7. Single sensor failure detection and identification. We have $y = Ax$, where $A \in \mathbb{R}^{m \times n}$ is known, and $x \in \mathbb{R}^n$ is to be found. Unfortunately, up to one sensor may have failed (but you don't know which one has failed, or even whether any has failed). You are given $\tilde{y}$ and not y, where $\tilde{y}$ is the same as y in all entries except, possibly, one (say, the kth entry). If all sensors are operating correctly, we have $y = \tilde{y}$. If the kth sensor fails, we have $\tilde{y}_i = y_i$ for all $i \neq k$.

The file `one_bad_sensor.m`, available on the course web site, defines A and $\tilde{y}$ (as `A` and `ytilde`). Determine which sensor has failed (or if no sensors have failed). You must explain your method, and submit your code.

For this exercise, you can use the matlab code `rank([F g])==rank(F)` to check if $g \in \text{range}(F)$. (We will see later a much better way to check if $g \in \text{range}(F)$.)

8. Projection matrices. A matrix $P \in \mathbb{R}^{n\times n}$ is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$.

a) Show that if P is a projection matrix then so is $I - P$.

b) Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

c) Suppose $A \in \mathbb{R}^{n\times k}$ is full rank, with $k \le n$. Show that $A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is a projection matrix.

d) If $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the point y in S closest to x is called the *projection of x on S*. Show that if P is a projection matrix, then $y = Px$ is the projection of x on $\text{range}(P)$. (Which is why such matrices are called projection matrices . . .)

9. Groups of equivalent statements. In the list below there are 11 statements about two square matrices A and B in $\mathbb{R}^{n\times n}$.

a) $\text{range}(B) \subseteq \text{range}(A)$.

b) there exists a matrix $Y \in \mathbb{R}^{n\times n}$ such that $B = YA$.

c) $AB = 0$.

d) $BA = 0$.

e) $\text{rank}(\begin{bmatrix} A & B \end{bmatrix}) = \text{rank}(A)$.

f) $\text{range}(A) \perp \text{null}(B^\mathsf{T})$.

g) $\text{rank}(\begin{bmatrix} A \\ B \end{bmatrix}) = \text{rank}(A)$.

h) $\text{range}(A) \subseteq \text{null}(B)$.

i) there exists a matrix $Z \in \mathbb{R}^{n\times n}$ such that $B = AZ$.

j) $\text{rank}(\begin{bmatrix} A & B \end{bmatrix}) = \text{rank}(B)$.

k) $\text{null}(A) \subseteq \text{null}(B)$.

Your job is to collect them into (the largest possible) groups of equivalent statements. Two statements are equivalent if each one implies the other. For example, the statement 'A is onto' is equivalent to 'null(A) $= \{0\}$' (when A is square, which we assume here), because every square matrix that is onto has zero nullspace, and vice versa. Two statements are not equivalent if there exist (real) square matrices A and B for which one holds, but the other does not. A group of statements is equivalent if any pair of statements in the group is equivalent.

We want *just* your answer, which will consist of lists of mutually equivalent statements; we do not need any justification.

Put your answer in the following specific form. List each group of equivalent statements on a line, in (alphabetic) order. Each new line should start with the first letter not listed above. For example, you might give your answer as

a, c, d, h

b, i

e

f, g, j, k.

This means you believe that statements a, c, d, and h are equivalent; statements b and i are equivalent; and statements f, g, j, and k are equivalent. You also believe that the first group of statements is not equivalent to the second, or the third, and so on.