1. **Quadratic extrapolation of a time series.** We are given a series $z$ up to time $t$. Using a quadratic model, we want to extrapolate, or predict, $z(t + 1)$ based on the three previous elements of the series, $z(t)$, $z(t - 1)$, and $z(t - 2)$. We’ll denote the predicted value of $z(t + 1)$ by $\hat{z}(t + 1)$. More precisely, you will find $\hat{z}(t + 1)$ as follows.

a) Find the quadratic function $f(\tau) = a_2 \tau^2 + a_1 \tau + a_0$ which satisfies $f(t) = z(t)$, $f(t - 1) = z(t - 1)$, and $f(t - 2) = z(t - 2)$. Then the extrapolated value is given by $\hat{z}(t + 1) = f(t + 1)$. Show that

$$\hat{z}(t + 1) = c \begin{bmatrix} z(t) \\ z(t - 1) \\ z(t - 2) \end{bmatrix},$$

where $c \in \mathbb{R}^{1 \times 3}$, and does not depend on $t$. In other words, the quadratic extrapolator is a linear function. Find $c$ explicitly.

b) Use the following Julia code to generate a time series $z$:

```julia
  t = collect(1:1000);
  z = 5*sin.(t/10 .+ 2) + 0.1 * sin.(t) + 0.1*sin.(2*t .- 5);
```

Use the quadratic extrapolation method from part (a) to find $\hat{z}(t)$ for $t = 4, \ldots, 1000$. Find the relative root-mean-square (RMS) error, which is given by

$$\left( \frac{1}{1000} \sum_{j=4}^{1000} (\hat{z}(j) - z(j))^2 \right)^{1/2} \left( \frac{1}{1000} \sum_{j=4}^{1000} z(j)^2 \right)^{1/2}.$$ 

2. **Price elasticity of demand.** The demand for $n$ different goods is a function of their prices: $q = f(p)$,

where $p$ is the price vector, $q$ is the demand vector, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the demand function. The current price and demand are denoted $p^*$ and $q^*$, respectively. Now suppose there is a small price change $\delta p$, so $p = p^* + \delta p$. This induces a change in demand, to $q \approx q^* + \delta q$, where

$$\delta q \approx Df(p^*) \delta p,$$

where $Df$ is the derivative or Jacobian of $f$, with entries

$$Df(p^*)_{ij} = \frac{\partial f_i}{\partial p_j}(p^*).$$

This is usually rewritten in term of the *elasticity matrix* $E$, with entries

$$E_{ij} = \frac{\partial f_i}{\partial p_j}(p^*) \frac{1/q_i^*}{1/p_j^*}.$$
so $E_{ij}$ gives the relative change in demand for good $i$ per relative change in price $j$. Defining the vector $y$ of relative demand changes, and the vector $x$ of relative price changes,

$$y_i = \frac{\delta q_i}{q_i^*}, \quad x_j = \frac{\delta p_j}{p_j^*},$$

we have the linear model $y = Ex$.

Here are the questions:

a) What is a reasonable assumption about the diagonal elements $E_{ii}$ of the elasticity matrix?

b) Goods $i$ and $j$ are called substitutes if they provide a similar service or other satisfaction (e.g., train tickets and bus tickets, cake and pie, etc.). They are called complements if they tend to be used together (e.g., automobiles and gasoline, left and right shoes, etc.). For each of these two generic situations, what can you say about $E_{ij}$ and $E_{ji}$?

c) Suppose the price elasticity of demand matrix for two goods is

$$E = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$ 

Describe the nullspace of $E$, and give an interpretation (in one or two sentences). What kind of goods could have such an elasticity matrix?

3. Halfspace. Suppose $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to $a$ than $b$ is a halfspace, i.e.:

$$\{ x \mid \|x - a\| \leq \|x - b\| \} = \{ x \mid c^T x \leq d \}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give $c$ and $d$ explicitly, and draw a picture showing $a$, $b$, $c$, and the halfspace.

4. Linearizing range measurements. Consider a single (scalar) measurement $y$ of the distance or range of $x \in \mathbb{R}^n$ to a fixed point or beacon at $a$, i.e., $y = \|x - a\|$.

a) Show that the linearized model near $x_0$ can be expressed as $\delta y = k^T \delta x$, where $k$ is the unit vector (i.e., with length one) pointing from $a$ to $x_0$. Derive this analytically, and also draw a picture (for $n = 2$) to demonstrate it.

b) Consider the error $e$ of the linearized approximation, i.e.,

$$e = \|x_0 + \delta x - a\| - \|x_0 - a\| - k^T \delta x.$$ 

The relative error of the approximation is given by $\eta = e/\|x_0 - a\|$. We know, of course, that the absolute value of the relative error is very small provided $\delta x$ is small. In many specific applications, it is possible and useful to make a stronger statement, for example, to derive a bound on how large the error can be. You will do that here. In fact you will prove that

$$0 \leq \eta \leq \frac{\alpha^2}{2}$$
where \( \alpha = \| \delta x \| / \| x_0 - a \| \) is the relative size of \( \delta x \). For example, for a relative displacement of \( \alpha = 1\% \), we have \( \eta \leq 0.00005 \), i.e., the linearized model is accurate to about 0.005\%. To prove this bound you can proceed as follows:

- Show that \( \eta = -1 + \sqrt{1 + \alpha^2 + 2\beta} - \beta \) where \( \beta = k^T \delta x / \| x_0 - a \| \).
- Verify that \( |\beta| \leq \alpha \).
- Consider the function \( g(\beta) = -1 + \sqrt{1 + \alpha^2 + 2\beta} - \beta \) with \( |\beta| \leq \alpha \). By maximizing and minimizing \( g \) over the interval \(-\alpha \leq \beta \leq \alpha \) show that \( 0 \leq \eta \leq \frac{\alpha^2}{2} \).

5. **Temperatures in a multi-core processor.** We are concerned with the temperature of a processor at two critical locations. These temperatures, denoted \( T = (T_1, T_2) \) (in degrees C), are affine functions of the power dissipated by three processor cores, denoted \( P = (P_1, P_2, P_3) \) (in W). We make 4 measurements. In the first, all cores are idling, and dissipate 10W. In the next three measurements, one of the processors is set to full power, 100W, and the other two are idling. In each experiment we measure and note the temperatures at the two critical locations.

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10W</td>
<td>10W</td>
<td>10W</td>
<td>27(^\circ)</td>
<td>29(^\circ)</td>
</tr>
<tr>
<td>100W</td>
<td>10W</td>
<td>10W</td>
<td>45(^\circ)</td>
<td>37(^\circ)</td>
</tr>
<tr>
<td>10W</td>
<td>100W</td>
<td>10W</td>
<td>41(^\circ)</td>
<td>49(^\circ)</td>
</tr>
<tr>
<td>10W</td>
<td>10W</td>
<td>100W</td>
<td>35(^\circ)</td>
<td>55(^\circ)</td>
</tr>
</tbody>
</table>

Suppose we operate all cores at the same power, \( p \). How large can we make \( p \), without \( T_1 \) or \( T_2 \) exceeding 70\(^\circ\)?

You must fully explain your reasoning and method, in addition to providing the numerical solution.

6. **Identifying a point on the unit sphere from spherical distances.** In this problem we consider the unit sphere in \( \mathbb{R}^n \), which is defined as the set of vectors with norm one: \( S^n = \{ x \in \mathbb{R}^n \mid \text{norm} x = 1 \} \). We define the spherical distance between two vectors on the unit sphere as the distance between them, measured along the sphere, i.e., as the angle between the vectors, measured in radians: If \( x, y \in S^n \), the spherical distance between them is

\[
\text{sphdist}(x, y) = \angle(x, y),
\]

where we take the angle as lying between 0 and \( \pi \). (Thus, the maximum distance between two points in \( S^n \) is \( \pi \), which occurs only when the two points \( x, y \) are antipodal, which means \( x = -y \).) Now suppose \( p_1, \ldots, p_k \in S^n \) are the (known) positions of some beacons on the unit sphere, and let \( x \in S^n \) be an unknown point on the unit sphere. We have exact measurements of the (spherical) distances between each beacon and the unknown point \( x \), i.e., we are given the numbers

\[
\rho_i = \text{sphdist}(x, p_i), \quad i = 1, \ldots, k.
\]
We would like to determine, without any ambiguity, the exact position of $x$, based on this information. Find the conditions on $p_1, \ldots, p_k$ under which we can unambiguously determine $x$, for any $x \in S^n$, given the distances $\rho_i$. You can give your solution algebraically, using any of the concepts used in class (e.g., nullspace, range, rank), or you can give a geometric condition (involving the vectors $p_i$). You must justify your answer.