

# Homework 1

EE 263 Stanford University Summer 2018

Due: July 05, 2018

- 1. A simple power control algorithm for a wireless network.** First some background. We consider a network of  $n$  transmitter/receiver pairs. Transmitter  $i$  transmits at power level  $p_i$  (which is positive). The path gain from transmitter  $j$  to receiver  $i$  is  $G_{ij}$  (which are all nonnegative, and  $G_{ii}$  are positive). The signal power at receiver  $i$  is given by  $s_i = G_{ii}p_i$ . The noise plus interference power at receiver  $i$  is given by

$$q_i = \sigma^2 + \sum_{j \neq i} G_{ij}p_j$$

where  $\sigma^2 > 0$  is the self-noise power of the receivers (assumed to be the same for all receivers). The *signal to interference plus noise ratio* (SINR) at receiver  $i$  is defined as  $S_i = s_i/q_i$ . For signal reception to occur, the SINR must exceed some threshold value  $\gamma$  (which is often in the range 3 – 10). Various *power control algorithms* are used to adjust the powers  $p_i$  to ensure that  $S_i \geq \gamma$  (so that each receiver can receive the signal transmitted by its associated transmitter). In this problem, we consider a simple power control update algorithm. The powers are all updated synchronously at a fixed time interval, denoted by  $t = 0, 1, 2, \dots$ . Thus the quantities  $p$ ,  $q$ , and  $S$  are discrete-time signals, so for example  $p_3(5)$  denotes the transmit power of transmitter 3 at time epoch  $t = 5$ . What we'd like is

$$S_i(t) = s_i(t)/q_i(t) = \alpha\gamma,$$

where  $\alpha > 1$  is an SINR safety margin (of, for example, one or two dB). Note that increasing  $p_i(t)$  (power of the  $i$ th transmitter) increases  $S_i$  but decreases all other  $S_j$ . A very simple power update algorithm is given by

$$p_i(t+1) = p_i(t)(\alpha\gamma/S_i(t)). \tag{1}$$

This scales the power at the next time step to be the power that would achieve  $S_i = \alpha\gamma$ , if the interference plus noise term were to stay the same. But unfortunately, changing the transmit powers also changes the interference powers, so it's not that simple! Finally, we get to the problem.

- a) Show that the power control algorithm can be expressed as a linear dynamical system with constant input, *i.e.*, in the form

$$p(t+1) = Ap(t) + b,$$

where  $A \in \mathbb{R}^{n \times n}$  and  $b \in \mathbb{R}^n$  are constant. Describe  $A$  and  $b$  explicitly in terms of  $\sigma$ ,  $\gamma$ ,  $\alpha$  and the components of  $G$ .

- b) *simulation* Simulate the power control algorithm, starting from various initial (positive) power levels. Use the problem data

$$G = \begin{bmatrix} 1 & .2 & .1 \\ .1 & 2 & .1 \\ .3 & .1 & 3 \end{bmatrix}, \quad \gamma = 3, \quad \alpha = 1.2, \quad \sigma = 0.1.$$

Plot  $S_i$  and  $p$  as a function of  $t$ , and compare it to the target value  $\alpha\gamma$ . Repeat for  $\gamma = 5$ . Comment briefly on what you observe. *Comment:* You'll understand what you see later in the course.

- 2. State equations for a linear mechanical system.** The equations of motion of a lumped mechanical system undergoing small motions can be expressed as

$$M\ddot{q} + D\dot{q} + Kq = f$$

where  $q(t) \in \mathbb{R}^k$  is the vector of deflections,  $M$ ,  $D$ , and  $K$  are the *mass*, *damping*, and *stiffness* matrices, respectively, and  $f(t) \in \mathbb{R}^k$  is the vector of externally applied forces. Assuming  $M$  is invertible, write linear system equations for the mechanical system, with state

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

input  $u = f$ , and output  $y = q$ .

- 3. Representing linear functions as matrix multiplication.** Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear. Show that there is a matrix  $A \in \mathbb{R}^{m \times n}$  such that for all  $x \in \mathbb{R}^n$ ,  $f(x) = Ax$ . (Explicitly describe how you get the coefficients  $A_{ij}$  from  $f$ , and then verify that  $f(x) = Ax$  for any  $x \in \mathbb{R}^n$ .) Is the matrix  $A$  that represents  $f$  unique? In other words, if  $\tilde{A} \in \mathbb{R}^{m \times n}$  is another matrix such that  $f(x) = \tilde{A}x$  for all  $x \in \mathbb{R}^n$ , then do we have  $\tilde{A} = A$ ? Either show that this is so, or give an explicit counterexample.

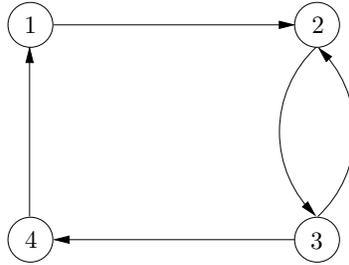
- 4. A mass subject to applied forces.** Consider a unit mass subject to a time-varying force  $f(t)$  for  $0 \leq t \leq n$ . Let the initial position and velocity of the mass both be zero. Suppose that the force has the form  $f(t) = x_j$  for  $j-1 \leq t < j$  and  $j = 1, \dots, n$ . Let  $y_1$  and  $y_2$  denote, respectively, the position and velocity of the mass at time  $t = n$ .

- a) Find the matrix  $A \in \mathbb{R}^{2 \times n}$  such that  $y = Ax$ .
- b) For  $n = 4$ , find a sequence of input forces  $x_1, \dots, x_n$  that moves the mass to position 1 with velocity 0 at time  $n$ .

- 5. Paths and cycles in a directed graph.** We consider a directed graph with  $n$  nodes. The graph is specified by its *node adjacency matrix*  $A \in \mathbb{R}^{n \times n}$ , defined as

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from node } j \text{ to node } i \\ 0 & \text{otherwise.} \end{cases}$$

Note that the edges are *oriented*, *i.e.*,  $A_{34} = 1$  means there is an edge from node 4 to node 3. For simplicity we do not allow self-loops, *i.e.*,  $A_{ii} = 0$  for all  $i$ ,  $1 \leq i \leq n$ . A simple example illustrating this notation is shown below.



The node adjacency matrix for this example is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this example, nodes 2 and 3 are connected in both directions, *i.e.*, there is an edge from 2 to 3 and also an edge from 3 to 2. A *path* of length  $l > 0$  from node  $j$  to node  $i$  is a sequence  $s_0 = j, s_1, \dots, s_l = i$  of nodes, with  $A_{s_{k+1}, s_k} = 1$  for  $k = 0, 1, \dots, l - 1$ . For example, in the graph shown above, 1, 2, 3, 2 is a path of length 3. A *cycle* of length  $l$  is a path of length  $l$ , with the same starting and ending node, with no repeated nodes other than the endpoints. In other words, a cycle is a sequence of nodes of the form  $s_0, s_1, \dots, s_{l-1}, s_0$ , with

$$A_{s_1, s_0} = 1, \quad A_{s_2, s_1} = 1, \quad \dots \quad A_{s_0, s_{l-1}} = 1,$$

and

$$s_i \neq s_j \text{ for } i \neq j, \quad i, j = 0, \dots, l - 1.$$

For example, in the graph shown above, 1, 2, 3, 4, 1 is a cycle of length 4. The rest of this problem concerns a specific graph, given in the file `directed_graph.m` on the course web site. For each of the following questions, you must give the answer explicitly (for example, enclosed in a box). You must also explain clearly how you arrived at your answer.

- What is the length of a shortest cycle? (Shortest means minimum length.)
- What is the length of a shortest path from node 13 to node 17? (If there are no paths from node 13 to node 17, you can give the answer as 'infinity'.)
- What is the length of a shortest path from node 13 to node 17, that *does not* pass through node 3?
- What is the length of a shortest path from node 13 to node 17, that *does* pass through node 9?
- Among all paths of length 10 that start at node 5, find the most common ending node.
- Among all paths of length 10 that end at node 8, find the most common starting node.
- Among all paths of length 10, find the most common pair of starting and ending nodes. In other words, find  $i, j$  which maximize the number of paths of length 10 from  $i$  to  $j$ .

**6. Color perception.** Human color perception is based on the responses of three different types of color light receptors, called *cones*. The three types of cones have different spectral-response characteristics, and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' responses as follows:

$$L_{\text{cone}} = \sum_{i=1}^{20} l_i p_i, \quad M_{\text{cone}} = \sum_{i=1}^{20} m_i p_i, \quad S_{\text{cone}} = \sum_{i=1}^{20} s_i p_i,$$

where  $p_i$  is the incident power in the  $i$ th wavelength band, and  $l_i$ ,  $m_i$  and  $s_i$  are nonnegative constants that describe the spectral responses of the different cones. The perceived color is a complex function of the three cone responses, *i.e.*, the vector  $(L_{\text{cone}}, M_{\text{cone}}, S_{\text{cone}})$ , with different cone response vectors perceived as different colors. (Actual color perception is a bit more complicated than this, but the basic idea is right.)

- a) *Metamers.* When are two light spectra,  $p$  and  $\tilde{p}$ , visually indistinguishable? (Visually identical lights with different spectral power compositions are called *metamers*.)
- b) *Visual color matching.* In a color matching problem, an observer is shown a test light, and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked to find a spectrum of the form

$$p_{\text{match}} = a_1 u + a_2 v + a_3 w,$$

where  $u$ ,  $v$ ,  $w$  are the spectra of the primary lights, and  $a_i$  are the intensities to be found, that is visually indistinguishable from a given test light spectrum  $p_{\text{test}}$ . Can this always be done? Discuss briefly.

- c) *Visual matching with phosphors.* A computer monitor has three phosphors,  $R$ ,  $G$ , and  $B$ . It is desired to adjust the phosphor intensities to create a color that looks like a reference test light. Find weights that achieve the match or explain why no such weights exist. The data for this problem is in `color_perception_data.json`, which contains the vectors `wavelength`, `B_phosphor`, `G_phosphor`, `R_phosphor`, `L_coefficients`, `M_coefficients`, `S_coefficients`, and `test_light`.
- d) *Effects of illumination.* An object's surface can be characterized by its reflectance (*i.e.*, the fraction of light it reflects) for each band of wavelengths. If the object is illuminated with a light spectrum characterized by  $I_i$ , and the reflectance of the object is  $r_i$  (which is between 0 and 1), then the reflected light spectrum is given by  $I_i r_i$ , where  $i = 1, \dots, 20$  denotes the wavelength band. Now consider two objects illuminated (at different times) by two different light sources, say an incandescent bulb and sunlight. Sally argues that if the two objects look identical when illuminated by a tungsten bulb, then they will look identical when illuminated by sunlight. Beth disagrees: she says that two objects can appear identical when illuminated by a tungsten bulb, but look different when lit by sunlight. Who is right? If Sally is right, explain why. If Beth is right give an example of two objects that appear identical under one light source and different under another. You can use the vectors `sunlight` and `tungsten` defined in the data file as the light sources.

*Remark.* Spectra, intensities, and reflectances are all nonnegative quantities, which the material of EE263 doesn't address. So just ignore this while doing this problem. These issues can be handled using the material of EE364a, however.

**7. Some properties of the product of two matrices.** For each of the following statements, either show that it is true, or give a (specific) counterexample.

- If  $AB$  is full rank then  $A$  and  $B$  are full rank.
- If  $A$  and  $B$  are full rank then  $AB$  is full rank.
- If  $A$  and  $B$  have zero nullspace, then so does  $AB$ .
- If  $A$  and  $B$  are onto, then so is  $AB$ .

You can assume that  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times p}$ . Some of the false statements above become true under certain assumptions on the dimensions of  $A$  and  $B$ . As a trivial example, all of the statements above are true when  $A$  and  $B$  are scalars, *i.e.*,  $n = m = p = 1$ . For each of the statements above, find conditions on  $n$ ,  $m$ , and  $p$  that make them true. Try to find the most general conditions you can. You can give your conditions as inequalities involving  $n$ ,  $m$ , and  $p$ , or you can use more informal language such as “ $A$  and  $B$  are both skinny.”