2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force $f(t)$ for $0 \leq t \leq n$. Let the initial position and velocity of the mass both be zero. Suppose that the force has the form $f(t) = x_j$ for $j - 1 \leq t < j$ and $j = 1, \ldots, n$. Let $y_1$ and $y_2$ denote, respectively, the position and velocity of the mass at time $t = n$.

a) Find the matrix $A \in \mathbb{R}^{2 \times n}$ such that $y = Ax$.

b) For $n = 4$, find a sequence of input forces $x_1, \ldots, x_n$ that moves the mass to position 1 with velocity 0 at time $n$.

2.120. Counting sequences in a language or code. We consider a language or code with an alphabet of $n$ symbols 1, 2, $\ldots$, $n$. A sentence is a finite sequence of symbols, $k_1, \ldots, k_m$ where $k_i \in \{1, \ldots, n\}$. A language or code consists of a set of sequences, which we will call the allowable sequences. A language is called Markov if the allowed sequences can be described by giving the allowable transitions between consecutive symbols. For each symbol we give a set of symbols which are allowed to follow the symbol. As a simple example, consider a Markov language with three symbols 1, 2, 3. Symbol 1 can be followed by 1 or 3; symbol 2 must be followed by 3; and symbol 3 can be followed by 1 or 2. The sentence 1132313 is allowable (i.e., in the language); the sentence 1132312 is not allowable (i.e., not in the language). To describe the allowed symbol transitions we can define a matrix $A \in \mathbb{R}^{n \times n}$ by

$$A_{ij} = \begin{cases} 
1 & \text{if symbol } i \text{ is allowed to follow symbol } j \\
0 & \text{if symbol } i \text{ is not allowed to follow symbol } j.
\end{cases}$$

a) Let $B = A^r$. Give an interpretation of $B_{ij}$ in terms of the language.

b) Consider the Markov language with five symbols 1, 2, 3, 4, 5, and the following transition rules:

- 1 must be followed by 2 or 3
- 2 must be followed by 2 or 5
- 3 must be followed by 1
- 4 must be followed by 4 or 2 or 5
- 5 must be followed by 1 or 3

Find the total number of allowed sentences of length 10. Compare this number to the simple code that consists of all sequences from the alphabet (i.e., all symbol transitions are allowed). In addition to giving the answer, you must explain how you solve the problem. Do not hesitate to use Julia.
2.150. Gradient of some common functions. Recall that the gradient of a differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), at a point \( x \in \mathbb{R}^n \), is defined as the vector

\[
\nabla f(x) = \begin{bmatrix}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{bmatrix},
\]

where the partial derivatives are evaluated at the point \( x \). The first order Taylor approximation of \( f \), near \( x \), is given by

\[
\hat{f}_{\text{tay}}(z) = f(x) + \nabla f(x)^T(z - x).
\]

This function is affine, i.e., a linear function plus a constant. For \( z \) near \( x \), the Taylor approximation \( \hat{f}_{\text{tay}} \) is very near \( f \). Find the gradient of the following functions. Express the gradients using matrix notation.

a) \( f(x) = a^T x + b \), where \( a \in \mathbb{R}^n \), \( b \in \mathbb{R} \).

b) \( f(x) = x^T Ax \), for \( A \in \mathbb{R}^{n \times n} \).

c) \( f(x) = x^T Ax \), where \( A = A^T \in \mathbb{R}^{n \times n} \). (Yes, this is a special case of the previous one.)

2.180. Paths and cycles in a directed graph. We consider a directed graph with \( n \) nodes. The graph is specified by its node adjacency matrix \( A \in \mathbb{R}^{n \times n} \), defined as

\[
A_{ij} = \begin{cases}
1 & \text{if there is an edge from node } j \text{ to node } i \\
0 & \text{otherwise.}
\end{cases}
\]

Note that the edges are oriented, i.e., \( A_{34} = 1 \) means there is an edge from node 4 to node 3. For simplicity we do not allow self-loops, i.e., \( A_{ii} = 0 \) for all \( i, 1 \leq i \leq n \). A simple example illustrating this notation is shown below.

```
1
\arrow{1} \rightarrow \node{2}
\arrow{4} \rightarrow \node{3}
```

The node adjacency matrix for this example is

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

In this example, nodes 2 and 3 are connected in both directions, i.e., there is an edge from 2 to 3 and also an edge from 3 to 2. A path of length \( l > 0 \) from node \( j \) to node \( i \) is a sequence
s_0 = j, s_1, \ldots, s_l = i of nodes, with A_{s_{k+1}, s_k} = 1 for k = 0, 1, \ldots, l - 1. For example, in the graph shown above, 1, 2, 3, 2 is a path of length 3. A cycle of length l is a path of length l, with the same starting and ending node, with no repeated nodes other than the endpoints. In other words, a cycle is a sequence of nodes of the form s_0, s_1, \ldots, s_{l-1}, s_0, with

$$A_{s_1, s_0} = 1, \quad A_{s_2, s_1} = 1, \quad \ldots \quad A_{s_0, s_{l-1}} = 1,$$

and

$$s_i \neq s_j \text{ for } i \neq j, \quad i, j = 0, \ldots, l - 1.$$  

For example, in the graph shown above, 1, 2, 3, 4, 1 is a cycle of length 4. The rest of this problem concerns a specific graph, given in the file \texttt{directed_graph.json} on the course website. For each of the following questions, you must give the answer explicitly (for example, enclosed in a box). You must also explain clearly how you arrived at your answer.

a) What is the length of a shortest cycle? (Shortest means minimum length.)

b) What is the length of a shortest path from node 13 to node 17? (If there are no paths from node 13 to node 17, you can give the answer as ‘infinity’.)

c) What is the length of a shortest path from node 13 to node 17, that does not pass through node 3?

d) What is the length of a shortest path from node 13 to node 17, that does pass through node 9?

e) Among all paths of length 10 that start at node 5, find the most common ending node.

f) Among all paths of length 10 that end at node 8, find the most common starting node.

g) Among all paths of length 10, find the most common pair of starting and ending nodes. In other words, find i, j which maximize the number of paths of length 10 from i to j.

\section*{3.240. Price elasticity of demand.}

The demand for \( n \) different goods is a function of their prices:

$$q = f(p),$$

where \( p \) is the price vector, \( q \) is the demand vector, and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the demand function. The current price and demand are denoted \( p^* \) and \( q^* \), respectively. Now suppose there is a small price change \( \delta p \), so \( p = p^* + \delta p \). This induces a change in demand, to \( q \approx q^* + \delta q \), where

$$\delta q \approx Df(p^*) \delta p,$$

where \( Df \) is the derivative or Jacobian of \( f \), with entries

$$Df(p^*)_{ij} = \frac{\partial f_i}{\partial p_j}(p^*).$$

This is usually rewritten in term of the \textit{elasticity matrix} \( E \), with entries

$$E_{ij} = \frac{\partial f_i}{\partial p_j}(p^*) \frac{1}{q_i^*} \frac{1}{p_j^*},$$
so $E_{ij}$ gives the relative change in demand for good $i$ per relative change in price $j$. Defining the vector $y$ of relative demand changes, and the vector $x$ of relative price changes,

$$y_i = \frac{\delta q_i}{q_i^*}, \quad x_j = \frac{\delta p_j}{p_j^*},$$

we have the linear model $y = Ex$.

Here are the questions:

a) What is a reasonable assumption about the diagonal elements $E_{ii}$ of the elasticity matrix?

b) Goods $i$ and $j$ are called substitutes if they provide a similar service or other satisfaction (e.g., train tickets and bus tickets, cake and pie, etc.). They are called complements if they tend to be used together (e.g., automobiles and gasoline, left and right shoes, etc.). For each of these two generic situations, what can you say about $E_{ij}$ and $E_{ji}$?

c) Suppose the price elasticity of demand matrix for two goods is

$$E = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$  

Describe the nullspace of $E$, and give an interpretation (in one or two sentences). What kind of goods could have such an elasticity matrix?