EE263 Homework 1

Fall 2023

2.50. Some linear functions associated with a convolution system. Suppose that u and y are scalar-valued discrete-time signals (*i.e.*, sequences) related via convolution:

$$y(k) = \sum_j h_j u(k-j), \quad k \in \mathbb{Z},$$

where $h_k \in \mathbb{R}$. You can assume that the convolution is *causal*, *i.e.*, $h_j = 0$ when $j < 0$.

a) *The input/output (Toeplitz) matrix.* Assume that $u(k) = 0$ for $k < 0$, and define

$$U = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix}, \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

Thus U and Y are vectors that give the first $N+1$ values of the input and output signals, respectively. Find the matrix T such that $Y = TU$. The matrix T describes the linear mapping from (a chunk of) the input to (a chunk of) the output. T is called the input/output or Toeplitz matrix (of size $N+1$) associated with the convolution system.

b) *The Hankel matrix.* Now assume that $u(k) = 0$ for $k > 0$ or $k < -N$ and let

$$U = \begin{bmatrix} u(0) \\ u(-1) \\ \vdots \\ u(-N) \end{bmatrix}, \quad Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}.$$

Here U gives the *past input* to the system, and Y gives (a chunk of) the resulting future output. Find the matrix H such that $Y = HU$. H is called the Hankel matrix (of size $N+1$) associated with the convolution system.

2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force $f(t)$ for $0 \le t \le n$. Let the initial position and velocity of the mass both be zero. Suppose that the force has the form $f(t) = x_j$ for $j-1 \le t < j$ and $j = 1, \ldots, n$. Let y_1 and y_2 denote, respectively, the position and velocity of the mass at time $t = n$.

a) Find the matrix $A \in \mathbb{R}^{2\times n}$ such that $y = Ax$.

b) For $n = 4$, find a sequence of input forces $x_1, \ldots, x_n$ that moves the mass to position 1 with velocity 0 at time n.

2.170. Affine functions. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if for any $x,\ y \in \mathbb{R}^n$ and any $\alpha,\ \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

(Without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.)

a) Suppose that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the function $f(x) = Ax + b$ is affine.

b) Now the converse: Show that any affine function f can be represented as $f(x) = Ax+b$, for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. (This representation is unique: for a given affine function f there is only one A and one b for which $f(x) = Ax + b$ for all x.)

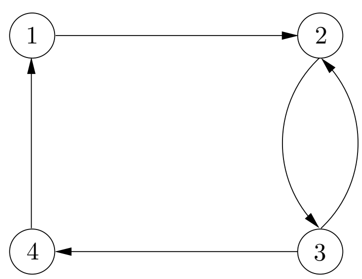
Hint. Show that the function $g(x) = f(x) - f(0)$ is linear.

You can think of an affine function as a linear function, plus an offset. In some contexts, affine functions are (mistakenly, or informally) called linear, even though in general they are not. (Example: $y = mx + b$ is described as 'linear' in US high schools.)

2.180. Paths and cycles in a directed graph. We consider a directed graph with n nodes. The graph is specified by its *node adjacency matrix* $A \in \mathbb{R}^{n\times n}$, defined as

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from node } j \text{ to node } i \\ 0 & \text{otherwise.} \end{cases}$$

Note that the edges are *oriented*, *i.e.*, $A_{34} = 1$ means there is an edge from node 4 to node 3. For simplicity we do not allow self-loops, *i.e.*, $A_{ii} = 0$ for all i, $1 \le i \le n$. A simple example illustrating this notation is shown below.



The node adjacency matrix for this example is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this example, nodes 2 and 3 are connected in both directions, *i.e.*, there is an edge from 2 to 3 and also an edge from 3 to 2. A *path* of length $l > 0$ from node j to node i is a sequence $s_0 = j, s_1, \ldots, s_l = i$ of nodes, with $A_{s_{k+1},s_k} = 1$ for $k = 0, 1, \ldots, l-1$. For example, in the graph shown above, $1, 2, 3, 2$ is a path of length 3. A *cycle* of length l is a path of length l, with the same starting and ending node, with no repeated nodes other than the endpoints. In other words, a cycle is a sequence of nodes of the form $s_0, s_1, \ldots, s_{l-1}, s_0$, with

$$A_{s_1,s_0} = 1, \quad A_{s_2,s_1} = 1, \quad \ldots \quad A_{s_0,s_{l-1}} = 1,$$

and

$$s_i \neq s_j \text{ for } i \neq j, \quad i, j = 0, \ldots, l-1.$$

For example, in the graph shown above, $1, 2, 3, 4, 1$ is a cycle of length 4. The rest of this problem concerns a specific graph, given in the file `directed_graph.json` on the course web site. For each of the following questions, you must give the answer explicitly (for example, enclosed in a box). You must also explain clearly how you arrived at your answer.

a) What is the length of a shortest cycle? (Shortest means minimum length.)

b) What is the length of a shortest path from node 13 to node 17? (If there are no paths from node 13 to node 17, you can give the answer as 'infinity'.)

c) What is the length of a shortest path from node 13 to node 17, that *does not* pass through node 3?

d) What is the length of a shortest path from node 13 to node 17, that *does* pass through node 9?

e) Among all paths of length 10 that start at node 5, find the most common ending node.

f) Among all paths of length 10 that end at node 8, find the most common starting node.

g) Among all paths of length 10, find the most common pair of starting and ending nodes. In other words, find q, r which maximize the number of paths of length 10 from q to r.

2.210. Express the following statements in matrix language. You can assume that all matrices mentioned have appropriate dimensions. Here is an example: "Every column of C is a linear combination of the columns of B" can be expressed as "$C = BF$ for some matrix F".

There can be several answers; one is good enough for us.

a) Suppose Z has n columns. For each i, row i of Z is a linear combination of rows $i, \ldots, n$ of Y.

b) W is obtained from V by permuting adjacent odd and even columns (*i.e.*, 1 and 2, 3 and 4, ...).

c) Each column of P makes an acute angle with each column of Q.

d) Each column of P makes an acute angle with the corresponding column of Q.

e) The first k columns of A are orthogonal to the remaining columns of A.

2.230. Population dynamics. An ecosystem consists of n species that interact (say, by eating other species, eating each other's food sources, eating each other's predators, and so on). We let $x(t) \in \mathbb{R}^n$ be the vector of deviations of the species populations (say, in thousands) from some equilibrium values (which don't matter here), in time period (say, month) t. In this model, time will take on the discrete values $t = 0, 1, 2, \ldots$. Thus $x_3(4) < 0$ means that the population of species 3 in time period 4 is below its equilibrium level. (It does not mean the population of species 3 is negative in time period 4.)

The population (deviations) follows a discrete-time linear dynamical system, which means that $x(t+1)$ is determined by $x(t)$. That is, we can compute the entire sequence $x(0), x(1), x(2), \ldots$ from $x(0)$ by applying the iteration

$$x(t+1) = Ax(t).$$

We refer to $x(0)$ as the *initial population perturbation*.

The questions below pertain to the specific case with $n = 10$ species, with matrix A given in `pop_dyn_data.json`.

a) Suppose the initial perturbation is $x(0) = e_4$ (meaning, we inject one thousand new creatures of species 4 into the ecosystem at $t = 0$). How long will it take to affect the other species populations? In other words, report a vector s, where s_i is the smallest t for which $x_i(t) \neq 0$. (We have $s_4 = 0$).

b) *Population control.* We can choose any initial perturbation that satisfies $|x_i(0)| \leq 1$ for each $i = 1, \ldots, 10$. (We achieve this by introducing additional creatures and/or hunting and fishing.) What initial perturbation $x(0)$ would you choose in order to maximize the population of species 1 at time $t = 10$? Explain your reasoning. Give the initial perturbation, and using your selected initial perturbation, give $x_1(10)$ and plot $x_1(t)$ versus t for $t = 0, \ldots, 40$.