Midterm exam

This is a 24 hour take-home midterm with 6 problems. Please turn it in at Bytes Cafe in the Packard building, 24 hours after you pick it up.

- You may use any books, notes, or computer programs (e.g., matlab), but you may not discuss the exam with others until Nov. 10, after everyone has taken the exam. The only exception is that you can ask the course staff for clarification, by emailing to the staff email address ee263-fall1617-staff@lists.stanford.edu. We’ve tried pretty hard to make the exam unambiguous and clear, so we’re unlikely to say much. Please do not post any exam related questions on Piazza.

- Since you have 24 hours, we expect your solutions to be legible, neat, and clear. Do not hand in your rough notes, and please try to simplify your solutions as much as you can. We will deduct points from solutions that are technically correct, but much more complicated than they need to be.

- Please check your email a few times during the exam, just in case we need to send out a clarification or other announcement. It’s unlikely we’ll need to do this, but you never know.

- Attach the official exam cover page (available when you pick up or drop off the exam) to your exam, and assemble your solutions to the problems in order, i.e., problem 1, problem 2, . . . , problem 6. Start each solution on a new page.

- Please make a copy of your exam before handing it in. We have never lost one, but it might occur.

- If a problem asks for some specific answers, make sure they are obvious in your solutions. You might put a box around the answers, so they stand out from the surrounding discussion, justification, plots, etc.

- When a problem involves some computation (say, using matlab), we do not want just the final answers. We want a clear discussion and justification of exactly what you did, the matlab source code that produces the result, and the final numerical result. Be sure to show us your verification that your computed solution satisfies whatever properties it is supposed to, at least up to numerical precision. For example, if you compute a vector \( x \) that is supposed to satisfy \( Ax = b \) (say), show us the matlab code that checks this, and the result. (This might be done by the matlab code \( \text{norm}(A*x-b) \); be sure to show us the result, which should be very small.) We will not check your numerical solutions for you, in cases where there is more than one solution.
• In the portion of your solutions where you explain the mathematical approach, you cannot refer to matlab operators, such as the backslash operator. (You can, of course, refer to inverses of matrices, or any other standard mathematical construct.)

• Some of the problems are described in a practical setting, such as population dynamics, or signal processing. You do not need to understand anything about the application area to solve these problems. We’ve taken special care to make sure all the information and math needed to solve the problem is given in the problem description.

• Some of the problems require you to download and run a matlab file to generate the data needed. These files can be found at the URL

   http://ee263.stanford.edu/mterm16_mfiles/Filename

where you should substitute the particular filename (given in the problem) for Filename. There are no links on the course web page pointing to these files, so you’ll have to type in the whole URL yourself.

• Please respect the honor code. Although we encourage you to work on homework assignments in small groups, you cannot discuss the midterm with anyone, with the exception of EE263 course staff, until Nov. 10, when everyone has taken it and the solutions are posted online.
1. Linear algebra done right

(a) Let \( A \in \mathbb{R}^{n \times n} \), if \( A^2 = A \) and \( \text{rank}(A) = n \), must \( A \) be the identity matrix? Prove or disprove it with a counterexample.

(b) Let \( A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times p} \). Is the following always true:

\[
\dim(\text{range}(AB)) = \dim(\text{range}(B)) - \dim(\text{null}(A) \cap \text{range}(B)),
\]

where \( \dim(V) \) gives the dimension of the vector space \( V \). Prove or disprove it with a counterexample.

Note: The operation \( \cap \) denotes intersection. That is, given any two sets \( A \) and \( B \), \( A \cap B \) is the set that contains elements both in \( A \) and in \( B \). Consequently, \( \text{null}(A) \cap \text{range}(B) \) in the problem means the set of all vectors that are in both the null space of \( A \) and the range of \( B \).

(c) Let \( u_1, \ldots, u_k \in \mathbb{R}^m \) and \( v_1, \ldots, v_k \in \mathbb{R}^n \) be column vectors. Consider the matrix \( M = \sum_{i=1}^k u_i v_i^T \). Compare the \( \text{rank}(M) \) with \( k \) and prove your claim.

(d) Let \( M \in \mathbb{R}^{m \times n} \) be a rank \( k \) matrix. Is it always possible to find column vectors \( u_1, \ldots, u_k \in \mathbb{R}^m \) and \( v_1, \ldots, v_k \in \mathbb{R}^n \) such that \( M = \sum_{i=1}^k u_i v_i^T \)? Prove or disprove (with a counter example) the claim.

Note: You should only use material covered in EE263 lectures up till this point.

(e) Let \( G = (V, E) \) be a simple and undirected graph, where \( V = \{1, 2, \ldots, n\} \) is the set of \( n \) vertices and \( E \) is the set of all edges. Let the matrix \( A \in \mathbb{R}^{n \times n} \) be the adjacency matrix of this graph \( G \), defined as follows:

\[
A_{ij} = \begin{cases} 
1, & (i, j) \text{ or } (j, i) \in E \\
0, & \text{otherwise}
\end{cases}
\]

(1)

Note that since the graph \( G \) being simple means that there is no self-loop, hence \( A_{ii} = 0 \) for each \( i \). Further, since the graph is undirected, \( A_{ij} = A_{ji}, \forall i, j \in V \).

A triangle in \( G \) is a set of three distinct vertices \( i, j, k \) where \( (i, j) \in E, (j, k) \in E, (k, i) \in E \). Let \( T \) be the total number of triangles in the graph \( G \). Prove the following statement:

\[
\text{Tr}(A^3) = 6T,
\]

where \( \text{Tr}(A) \) gives the trace of the square matrix \( A \in \mathbb{R}^{n \times n} \), which is equal to the sum of all its diagonal entries (i.e. \( \text{Tr}(A) = \sum_{i=1}^n A_{ii} \)).

Note: To help you better understand the problem, consider the below example graph:
where the vertex set is $V = \{1, 2, 3, 4\}$, and the edge set is $E = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4)\}$. The adjacency matrix is:

$$A = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}$$

You can easily check that in this case the number of triangles is 2 and $\text{Tr}(A^3) = 12$. As a final hint, in Review Session 4, we discussed what each entry in $A^2$ means. Think about what each entry on the diagonal of $A^3$ gives you.
2. Population dynamics

An ecosystem consists of \( n \) species that interact (say, by eating other species, eating each other’s food sources, eating each other’s predators, and so on). We let \( x(t) \in \mathbb{R}^n \) be the vector of deviations of the species populations (say, in thousands) from some equilibrium values (which don’t matter here), in time period (say, month) \( t \). In this model, time will take on the discrete values \( t = 0, 1, 2, \ldots \). Thus \( x_3(4) < 0 \) means that the population of species 3 in time period 4 is below its equilibrium level. (It does not mean the population of species 3 is negative in time period 4.) The population (deviations) follows a discrete-time linear dynamical system:

\[
x(t + 1) = Ax(t).
\]

We refer to \( x(0) \) as the initial population deviation.

The questions below pertain to the specific case with \( n = 10 \) species, with matrix \( A \) given in pop_dyn_data.m.

(a) Suppose the initial deviation is \( x(0) = e_4 \) (meaning, we inject one thousand new creatures of species 4 into the ecosystem at \( t = 0 \)). How long will it take to affect the other species populations? In other words, report a vector \( s \), where \( s_i \) is the smallest \( t \) for which \( x_i(t) \neq 0 \). (We have \( s_4 = 0 \)).

**Note:** We are not looking for an analytical answer here. You can answer this by running a matlab simulation.

(b) Population control. We can choose any initial deviation that satisfies \( |x_i(0)| \leq 1 \) for each \( i = 1, \ldots, 10 \). (We achieve this by introducing additional creatures and/or hunting and fishing.) What initial deviation \( x(0) \) would you choose in order to maximize the population of species 1 at \( t = 10 \)? Explain your reasoning. Give the initial deviation, and using your selected initial deviation, give \( x_1(10) \) and plot \( x_1(t) \) versus \( t \) for \( t = 0, \ldots, 40 \).
3. Network design

In this problem, you will be playing the role of a network engineer. You are given a fully connected network and your job is to set the bandwidths of the links in the network such that the state of the network gets as close as possible to a desired state, at a given point in time. Before we setup the problem, let’s define some network terminology.

We consider a simple model of a communication network in which message packets of unit-length are carried between the nodes. Assume that you are given a 2-dimensional communication network formed by the set of nodes \( N = \{(i, j) : i, j \in \mathbb{Z}, 1 \leq i \leq m, 1 \leq j \leq n\} \). Note that each node \((i, j)\) is indexed by two positive integers. It is your job to set up the links (i.e. connection wires) between the nodes. In this regard, for each node \((i, j)\) you need to allocate the bandwidths \( b_{(i,j) \to (k,l)} \), \((k, l) \in N\), \((k, l) \neq (i, j)\), for the wires connecting node \((i, j)\) to other nodes \((k, l)\). Note that the link that goes from node \((i, j)\) to node \((k, l)\) is distinct from the one going from node \((k, l)\) to node \((i, j)\). Also note that what we mean by \((k, l) \neq (i, j)\) is that either \(k \neq i\), or \(l \neq j\) or both. The bandwidths \( b_{(i,j) \to (k,l)} \) determine the out-flow dividing coefficients at each node, meaning that when we have some out-flow at node \((i, j)\), the following fraction of the out-flow will be carried to node \((k, l)\)

\[
\phi_{(i,j) \to (k,l)} = \frac{b_{(i,j) \to (k,l)}}{\sum_{(p,q) \in N, (p,q) \neq (i,j)} b_{(i,j) \to (p,q)}}.
\]

(We assume that for all \((i, j) \in N\), \(\sum_{(p,q) \in N, (p,q) \neq (i,j)} b_{(i,j) \to (p,q)} \neq 0\).) We study this network as a discrete-time system for \(0 \leq t \leq 2\). Let \(x_{(i,j)}(t)\) denote the total out-flow from node \((i, j)\) at time \(t\). In addition, let \(u_{(i,j)}(t)\) denote the known in-flow entering the node \((i, j)\) from outside the network at time \(t\). One simple model for the network as it evolves toward a steady state through time is as follows:

\[
x_{(i,j)}(t + 1) = u_{(i,j)}(t) + \sum_{(k,l) \in N, (k,l) \neq (i,j)} \phi_{(k,l) \to (i,j)} x_{(k,l)}(t), \quad 0 \leq t \leq 2.
\]

Assume that at \(t = 0\) there is no information out-flow in the network, which means that \(x_{(i,j)}(0) = 0\) for all \((i, j) \in N\). Also, assume that at each time \(t\), the in-flow entering the network is determined by known matrix \(U(t) \in \mathbb{R}^{m \times n}\), such that \(u_{(i,j)}(t) = U_{ij}(t)\).

We define the state of the network at time \(t\) using a matrix \(X(t) \in \mathbb{R}^{m \times n}\) for which \(X_{ij}(t) = x_{(i,j)}(t)\). Finally, the problem of interest is as follows: as the network engineer, you want the state of the system at time \(t = 2\) to be as close as possible to some desired state \(Y \in \mathbb{R}^{m \times n}\) in the sense of minimizing the following cost function:

\[
\|X(2) - Y\|_F^2,
\]

where \(\|X\|_F\) is the Frobenius norm of a matrix and is defined as:

\[
\|X\|_F = \left(\sum_{i,j} X_{ij}^2\right)^{1/2}.
\]
Remark: In reality, the in-flows and out-flows of the network should all be non-negative quantities. However, to simplify things, you need not enforce this constraint when solving this problem. In other words, it is ok if your solution produces negative bandwidths for some of the links.

(a) Assume that you are given a set of bandwidths \( \{ b^*_{(i,j)\to(k,l)} : (i, j), (k, l) \in N \} \) that minimizes the cost function (2). Determine whether this set of bandwidths would be the unique minimizer of our cost function, or we can find other solutions as well?

**Hint:** What would happen if you scale the optimal bandwidths \( b^*_{(i,j)\to(k,l)} \)?

(b) Using the material covered in EE263 so far, propose a method to find the bandwidths \( b_{(i,j)\to(k,l)} \) that minimize the cost function (2). Explicitly mention any assumptions that you make about the matrices that you use in your solution. Comment on whether or not your method gives a unique solution for optimal bandwidths \( b_{(i,j)\to(k,l)} \) and why.

**Hint:** Note that your cost function (2) is not the standard least-squares cost function. However, there is a simple connection between the two. You can start by rearranging matrices \( X(t) \) and \( Y \) into vectors. You will also need to somehow rearrange the bandwidths \( b_{(i,j)\to(k,l)} \) into a vector. Now this is a little bit trickier. Think of how many entries this vector will have? To make this a little easier you can define \( b_{(i,j)\to(i,j)} = 0 \) for all \( (i, j) \in N \) and include them as part of the bandwidths \( b_{(i,j)\to(k,l)} \). Now try to rewrite you cost function (2) in terms of these new vectorized variables and see if it looks more like a standard least-squares cost function. Yes we know, the relationship between \( \phi_{(i,j)\to(k,l)} \) and \( b_{(i,j)\to(k,l)} \) is making the math kind of messy. That is why we added part (a) to this problem. It is an indirect hint on how to make the math simpler. Finally, remember you need to somehow ensure that in your final solution \( b_{(i,j)\to(i,j)} = 0 \) for all \( (i, j) \in N \). How about adding some constraints to your minimization problem?

(c) Implement your method on the data given in `network_design_data.m`. In this m-file, the matrices \( U_0 \in \mathbb{R}^{m \times n} \) and \( U_1 \in \mathbb{R}^{m \times n} \) are the in-flows at times \( t = 0 \), and \( t = 1 \), respectively. The matrix \( Y \in \mathbb{R}^{m \times n} \) is the desired state at time \( t = 2 \). You only need to include your code and report the minimum value of cost function (2) for the given data.

**Hint:** The matlab function `pinv` can be used to find the pseudoinverse of a (not necessarily full-rank) matrix.
4. **Grab life by the Llama**

You are the owner of an alpine mountaineering company, who offer tours with guide llamas. You want to show your customers as much of the Llama Adventure WonderLand (L.A.W.L), but you don’t want to over exert your animals. We can model a llama as a first order linear dynamical system:

\[ x(t + 1) = Ax(t) + Bu(t) \]

\[
\begin{bmatrix}
1 & 0 & 0.1 & 0 \\
0 & 1 & 0 & 0.1 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}
\]

where \( x(t) \in \mathbb{R}^4 \) is the state vector consisting of the llamas position and velocity, and the system input \( u(t) \in \mathbb{R}^2 \) is the velocity control effort.

\[
x(t) = \begin{bmatrix} \text{pos}_x(t) \\ \text{pos}_y(t) \\ \text{vel}_x(t) \\ \text{vel}_y(t) \end{bmatrix} \quad u(t) = \begin{bmatrix} u_x(t) \\ u_y(t) \end{bmatrix}
\]

Our goal is minimize the overall effort put into controlling the llama between times 0 and \( T \) by minimizing the following cost function for a given \( T \).

\[
J = \sum_{t=0}^{T} \| u(t) \|^2
\]

We also want to hit a set of positional waypoints \( \{w_1, ..., w_n\}, w_i \in \mathbb{R}^2 \) at target times \( \{k_1, ..., k_n\} \). Hence, the minimization problem has the following constraints:

\[
w_i = Hx(k_i) \quad i = 1, ..., n
\]

where \( H \) is a matrix that when left multiplied by \( x \) returns the first two entries, since the waypoints are only positions.

\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Finally we get to the problem.

(a) Determine a single matrix equation that relates the control inputs \( u(t) \) for \( t = 0, ..., T \) to the waypoints \( w_i \) for \( i = 1, ..., n \). Assume you are given an initial state \( x(0) = x_0 \) and that the final time is \( T = k_n - 1 \).

**Hint:** expand out the dynamics for \( x(1), x(2), ... \) until you find a pattern. Your final equation should express \( w_i \)'s as an affine function of \( u(t) \) for \( t = 0, ..., T \) (remember, an affine function is a linear function plus a constant).
(b) Using the matrix equation of part (a), formulate the problem as a standard least-norm problem and find the optimal sequence of control inputs $u(t)$ that minimizes the cost function subject to the constraints. Leave your answer in terms of the problem data $(A, B, H, x_0, w_i, k_i)$. Feel free to introduce new variables to simplify your math.

(c) Implement your solution with the data in the file `llama_data.m`. Each column of $w$ gives the $(\text{pos}_x, \text{pos}_y)$ position waypoint for the corresponding time in $k$. Report the following plots:

- The trajectory of the llama with the waypoints clearly marked
- Each $x$ and $y$ velocity of the llama as a function of time

Also report your final minimized cost $J$.

(d) What are the conditions on matrices $A$ and $B$ that must hold in order to guarantee that the llama passes through all the waypoints at the target times? We are looking for a mathematical argument using what you have learned in EE263 up until this point.
Flux balance analysis in systems biology

Flux balance analysis is based on a very simple model of the reactions going on in a cell, keeping track only of the gross conservation of various chemical species (metabolites) within the cell.

We focus on \( m \) metabolites in a cell, labeled \( M_1, \ldots, M_m \). There are \( n \) (reversible) reactions going on, labeled \( R_1, \ldots, R_n \), with reaction rates \( v_1, \ldots, v_n \in \mathbb{R} \). A positive value of \( v_i \) means the reaction proceeds in the given direction, while a negative value of \( v_i \) means the reaction proceeds in the reverse direction. Each reaction has a (known) stoichiometry, which tells us the rate of consumption and production of the metabolites per unit of reaction rate. The stoichiometry data is given by the stoichiometry matrix \( S \in \mathbb{R}^{m \times n} \), defined as follows: \( S_{ij} \) is the rate of production of \( M_i \) due to unit reaction rate \( v_j = 1 \). Here we consider consumption of a metabolite as negative production; so \( S_{ij} = -2 \), for example, means that reaction \( R_j \) causes metabolite \( M_i \) to be consumed at a rate \( 2v_j \). If \( v_j \) is negative, then metabolite \( M_i \) is produced at the rate \( 2|v_j| \).

As an example, suppose reaction \( R_1 \) has the form \( M_1 \rightarrow M_2 + 2M_3 \). The consumption rate of \( M_1 \), due to this reaction, is \( v_1 \); the production rate of \( M_2 \) is \( v_1 \); and the production rate of \( M_3 \) is \( 2v_1 \). (The reaction \( R_1 \) has no effect on metabolites \( M_4, \ldots, M_m \).) This corresponds to a first column of \( S \) of the form \((-1, 1, 2, 0, \ldots, 0)\).

Reactions are also used to model flow of metabolites into and out of the cell. For example, suppose that reaction \( R_2 \) corresponds to the flow of metabolite \( M_1 \) into the cell, with \( v_2 \) giving the flow rate. (When \( v_2 < 0 \), it means that \( |v_2| \) is the flow rate of the metabolite out of the cell.) This corresponds to a second column of \( S \) of the form \((1, 0, \ldots, 0)\).

The last reaction, \( R_n \), corresponds to biomass creation, or cell growth, so the reaction rate \( v_n \) is the cell growth rate. The last column of \( S \) gives the amounts of metabolites used (when the entry is negative) or created (when positive) per unit of cell growth rate.

Since our reactions include metabolites entering or leaving the cell, as well as those converted to biomass within the cell, we have conservation of the metabolites, which can be expressed as the flux balance equation \( Sv = 0 \).

Finally, we consider the effect of knocking out a gene. For simplicity, we’ll assume that reactions \( 1, \ldots, n-1 \) are each controlled by an associated gene, \( i.e., \) gene \( G_k \) controls reaction \( R_k \). Knocking out \( G_k \) has the effect of setting the associated reaction rate to zero.

Finally, we get to the point of all this. Suppose there is no \( v \in \mathbb{R}^n \) that satisfies

\[
Sv = 0, \quad v_k = 0, \quad v_n > 0.
\]

This means there are no reaction rates consistent with cell growth, flux balance, and the gene knockout (remember, \( v_n \) is the cell growth rate). In this case, we predict that knocking out gene \( G_k \) will kill the cell, and call gene \( G_k \) an essential gene.
(a) Explain how to find all essential genes, given the stoichiometry matrix $S$. You can use any concepts from the class, e.g., range, nullspace, least-squares.

(b) Carry out your method for the problem data given in fba_data.m. List all essential genes.