Lecture 15
Symmetric matrices, quadratic forms, matrix norm, and SVD

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices
- norm of a matrix
- singular value decomposition

Eigenvalues of symmetric matrices

suppose $A \in \mathbf{R}^{n \times n}$ is symmetric, *i.e.*, $A = A^T$

fact: the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbf{C}^n$

then

$$\overline{v}^T A v = \overline{v}^T (Av) = \lambda \overline{v}^T v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\overline{v}^T A v = \overline{(Av)}^T v = \overline{(\lambda v)}^T v = \overline{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have $\lambda = \overline{\lambda}$, *i.e.*, $\lambda \in \mathbf{R}$ (hence, can assume $v \in \mathbf{R}^n$)

Eigenvectors of symmetric matrices

fact: there is a set of orthonormal eigenvectors of A, *i.e.*, $q_1, \ldots, q_n$ s.t. $Aq_i = \lambda_i q_i$, $q_i^T q_j = \delta_{ij}$

in matrix form: there is an orthogonal Q s.t.

$$Q^{-1} A Q = Q^T A Q = \Lambda$$

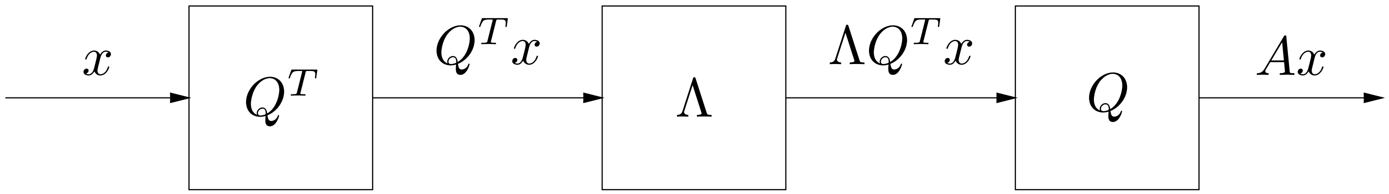
hence we can express A as

$$A = Q \Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

in particular, q_i are both left and right eigenvectors

Interpretations

$A = Q\Lambda Q^T$



linear mapping $y = Ax$ can be decomposed as

- resolve into q_i coordinates
- scale coordinates by λ_i
- reconstitute with basis q_i

or, geometrically,

- rotate by Q^T
- diagonal real scale ('dilation') by Λ
- rotate back by Q

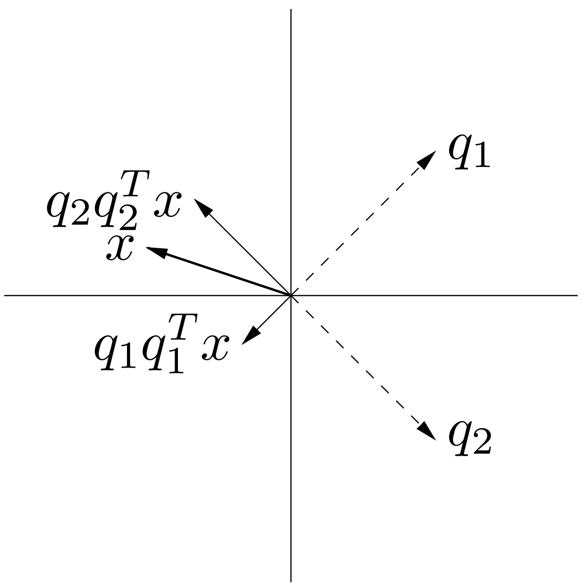
decomposition

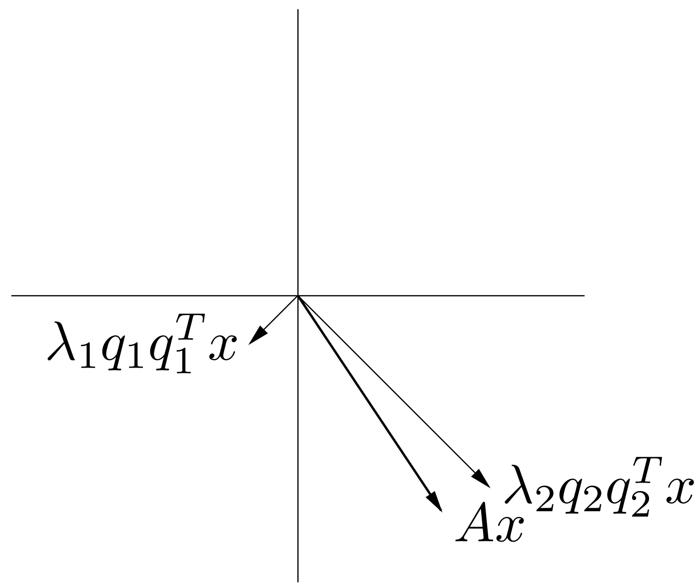
$$A = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

expresses A as linear combination of 1-dimensional projections

example:

$$
\begin{aligned}
A &= \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \\
&= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^T
\end{aligned}
$$





proof (case of λ_i distinct)

since λ_i distinct, can find $v_1, \ldots, v_n$, a set of linearly independent eigenvectors of A:

$$Av_i = \lambda_i v_i, \qquad \|v_i\| = 1$$

then we have

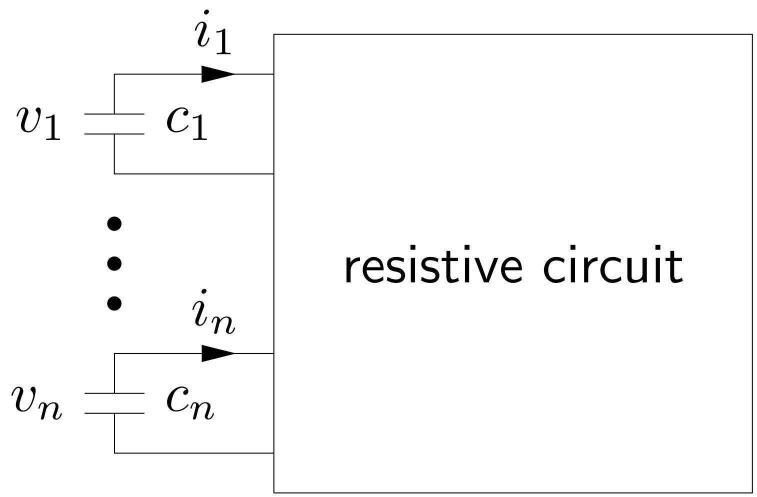
$$v_i^T(Av_j) = \lambda_j v_i^T v_j = (Av_i)^T v_j = \lambda_i v_i^T v_j$$

so $(\lambda_i - \lambda_j) v_i^T v_j = 0$

for $i \neq j$, $\lambda_i \neq \lambda_j$, hence $v_i^T v_j = 0$

- in this case we can say: eigenvectors *are* orthogonal
- in general case (λ_i not distinct) we must say: eigenvectors *can be chosen* to be orthogonal

Example: RC circuit



$$c_k \dot{v}_k = -i_k, \qquad i = Gv$$

$G = G^T \in \mathbf{R}^{n\times n}$ is conductance matrix of resistive circuit

thus $\dot{v} = -C^{-1}Gv$ where $C = \mathbf{diag}(c_1, \ldots, c_n)$

note $-C^{-1}G$ is not symmetric

use state $x_i = \sqrt{c_i} v_i$, so

$$\dot{x} = C^{1/2}\dot{v} = -C^{-1/2}GC^{-1/2}x$$

where $C^{1/2} = \mathbf{diag}(\sqrt{c_1}, \ldots, \sqrt{c_n})$

we conclude:

- eigenvalues $\lambda_1, \ldots, \lambda_n$ of $-C^{-1/2}GC^{-1/2}$ (hence, $-C^{-1}G$) are real
- eigenvectors q_i (in x_i coordinates) can be chosen orthogonal
- eigenvectors in voltage coordinates, $s_i = C^{-1/2}q_i$, satisfy

$$-C^{-1}Gs_i = \lambda_i s_i, \qquad s_i^T C s_i = \delta_{ij}$$

Quadratic forms

a function $f : \mathbf{R}^n \to \mathbf{R}$ of the form

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

is called a *quadratic form*

in a quadratic form we may as well assume $A = A^T$ since

$$x^T A x = x^T ((A + A^T)/2) x$$

($(A + A^T)/2$ is called the *symmetric part* of A)

uniqueness: if $x^T A x = x^T B x$ for all $x \in \mathbf{R}^n$ and $A = A^T$, $B = B^T$, then $A = B$

Examples

- $\|Bx\|^2 = x^T B^T Bx$
- $\sum_{i=1}^{n-1}(x_{i+1} - x_i)^2$
- $\|Fx\|^2 - \|Gx\|^2$

sets defined by quadratic forms:

- $\{ x \mid f(x) = a \}$ is called a *quadratic surface*
- $\{ x \mid f(x) \leq a \}$ is called a *quadratic region*

Inequalities for quadratic forms

suppose $A = A^T$, $A = Q\Lambda Q^T$ with eigenvalues sorted so $\lambda_1 \geq \cdots \geq \lambda_n$

$$\begin{aligned} x^T A x &= x^T Q \Lambda Q^T x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= \sum_{i=1}^{n} \lambda_i (q_i^T x)^2 \\ &\leq \lambda_1 \sum_{i=1}^{n} (q_i^T x)^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

$i.e.$, we have $x^T A x \leq \lambda_1 x^T x$

similar argument shows $x^T A x \geq \lambda_n \|x\|^2$, so we have

$$\lambda_n x^T x \leq x^T A x \leq \lambda_1 x^T x$$

sometimes λ_1 is called $\lambda_{\max}$, λ_n is called $\lambda_{\min}$

note also that

$$q_1^T A q_1 = \lambda_1 \|q_1\|^2, \qquad q_n^T A q_n = \lambda_n \|q_n\|^2,$$

so the inequalities are tight

Positive semidefinite and positive definite matrices

suppose $A = A^T \in \mathbf{R}^{n \times n}$

we say A is *positive semidefinite* if $x^T Ax \geq 0$ for all x

- denoted $A \geq 0$ (and sometimes $A \succeq 0$)
- $A \geq 0$ if and only if $\lambda_{\min}(A) \geq 0$, $i.e.$, all eigenvalues are nonnegative
- **not** the same as $A_{ij} \geq 0$ for all i, j

we say A is *positive definite* if $x^T Ax > 0$ for all $x \neq 0$

- denoted $A > 0$
- $A > 0$ if and only if $\lambda_{\min}(A) > 0$, $i.e.$, all eigenvalues are positive

Matrix inequalities

- we say A is *negative semidefinite* if $-A \geq 0$
- we say A is *negative definite* if $-A > 0$
- otherwise, we say A is *indefinite*

matrix inequality: if $B = B^T \in \mathbf{R}^n$ we say $A \geq B$ if $A - B \geq 0$, $A < B$ if $B - A > 0$, etc.

for example:

- $A \geq 0$ means A is positive semidefinite
- $A > B$ means $x^T A x > x^T B x$ for all $x \neq 0$

many properties that you'd guess hold actually do, $e.g.$,

- if $A \geq B$ and $C \geq D$, then $A + C \geq B + D$
- if $B \leq 0$ then $A + B \leq A$
- if $A \geq 0$ and $\alpha \geq 0$, then $\alpha A \geq 0$
- $A^2 \geq 0$
- if $A > 0$, then $A^{-1} > 0$

matrix inequality is only a *partial order*: we can have

$$A \not\geq B, \qquad B \not\geq A$$

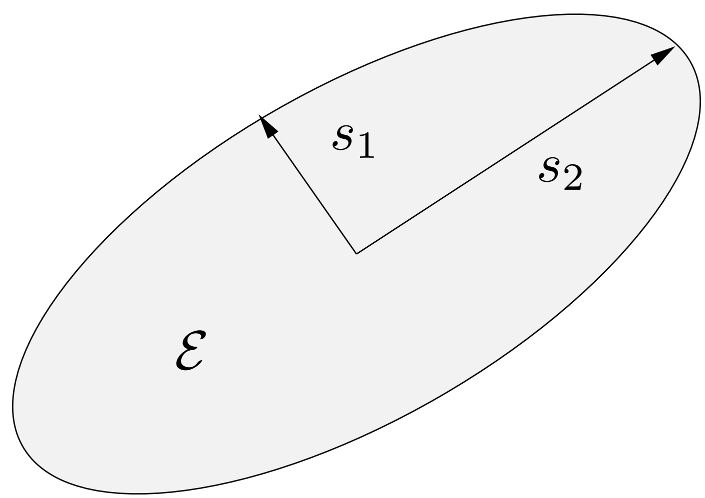
(such matrices are called *incomparable*)

Ellipsoids

if $A = A^T > 0$, the set

$$\mathcal{E} = \{ \ x \mid x^T A x \leq 1 \ \}$$

is an *ellipsoid* in $\mathbf{R}^n$, centered at 0



semi-axes are given by $s_i = \lambda_i^{-1/2} q_i$, *i.e.*:

- eigenvectors determine directions of semiaxes
- eigenvalues determine lengths of semiaxes

note:

- in direction q_1, $x^T A x$ is *large*, hence ellipsoid is *thin* in direction q_1
- in direction q_n, $x^T A x$ is *small*, hence ellipsoid is *fat* in direction q_n
- $\sqrt{\lambda_{\max}/\lambda_{\min}}$ gives maximum *eccentricity*

if $\tilde{\mathcal{E}} = \{ x \mid x^T B x \leq 1 \}$, where $B > 0$, then $\mathcal{E} \subseteq \tilde{\mathcal{E}} \iff A \geq B$

Gain of a matrix in a direction

suppose $A \in \mathbf{R}^{m \times n}$ (not necessarily square or symmetric)

for $x \in \mathbf{R}^n$, $\|Ax\|/\|x\|$ gives the *amplification factor* or *gain* of A in the direction x

obviously, gain varies with direction of input x

questions:

- what is maximum gain of A
  (and corresponding maximum gain direction)?
- what is minimum gain of A
  (and corresponding minimum gain direction)?
- how does gain of A vary with direction?

Matrix norm

the maximum gain

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

is called the *matrix norm* or *spectral norm* of A and is denoted $\|A\|$

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^T A^T A x}{\|x\|^2} = \lambda_{\max}(A^T A)$$

so we have $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$

similarly the minimum gain is given by

$$\min_{x \neq 0} \|Ax\|/\|x\| = \sqrt{\lambda_{\min}(A^T A)}$$

note that

- $A^T A \in \mathbf{R}^{n \times n}$ is symmetric and $A^T A \geq 0$ so $\lambda_{\min},\ \lambda_{\max} \geq 0$

- 'max gain' input direction is $x = q_1$, eigenvector of $A^T A$ associated with $\lambda_{\max}$

- 'min gain' input direction is $x = q_n$, eigenvector of $A^T A$ associated with $\lambda_{\min}$

example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\begin{aligned} A^T A &= \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} \\ &= \begin{bmatrix} 0.620 & 0.785 \\ 0.785 & -0.620 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} 0.620 & 0.785 \\ 0.785 & -0.620 \end{bmatrix}^T \end{aligned}$$

then $\|A\| = \sqrt{\lambda_{\max}(A^T A)} = 9.53$:

$$\left\| \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2.18 \\ 4.99 \\ 7.78 \end{bmatrix} \right\| = 9.53$$

min gain is $\sqrt{\lambda_{\min}(A^TA)} = 0.514$:

$$\left\| \begin{bmatrix} 0.785 \\ -0.620 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} 0.785 \\ -0.620 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0.46 \\ 0.14 \\ -0.18 \end{bmatrix} \right\| = 0.514$$

for all $x \neq 0$, we have

$$0.514 \leq \frac{\|Ax\|}{\|x\|} \leq 9.53$$

Properties of matrix norm

- consistent with vector norm: matrix norm of $a \in \mathbf{R}^{n\times 1}$ is $\sqrt{\lambda_{\max}(a^Ta)} = \sqrt{a^Ta}$
- for any x, $\|Ax\| \leq \|A\|\|x\|$
- scaling: $\|aA\| = |a|\|A\|$
- triangle inequality: $\|A+B\| \leq \|A\| + \|B\|$
- definiteness: $\|A\| = 0 \quad \Leftrightarrow \quad A = 0$
- norm of product: $\|AB\| \leq \|A\|\|B\|$

Singular value decomposition

more complete picture of gain properties of A given by *singular value decomposition* (SVD) of A:

$$A = U\Sigma V^T$$

where

- $A \in \mathbf{R}^{m\times n}$, $\mathbf{Rank}(A) = r$
- $U \in \mathbf{R}^{m\times r}$, $U^TU = I$
- $V \in \mathbf{R}^{n\times r}$, $V^TV = I$
- $\Sigma = \mathbf{diag}(\sigma_1, \ldots, \sigma_r)$, where $\sigma_1 \geq \cdots \geq \sigma_r > 0$

with $U = [u_1 \cdots u_r]$, $V = [v_1 \cdots v_r]$,

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

- σ_i are the (nonzero) *singular values* of A
- v_i are the *right* or *input singular vectors* of A
- u_i are the *left* or *output singular vectors* of A

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^2 V^T$$

hence:

- v_i are eigenvectors of $A^T A$ (corresponding to nonzero eigenvalues)
- $\sigma_i = \sqrt{\lambda_i(A^T A)}$ (and $\lambda_i(A^T A) = 0$ for $i > r$)
- $\|A\| = \sigma_1$

similarly,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma^2 U^T$$

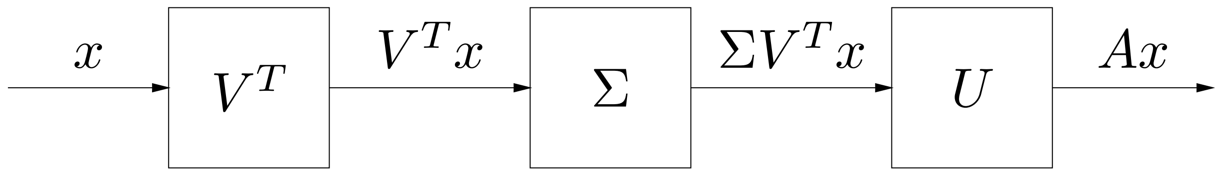
hence:

- u_i are eigenvectors of AA^T (corresponding to nonzero eigenvalues)
- $\sigma_i = \sqrt{\lambda_i(AA^T)}$ (and $\lambda_i(AA^T) = 0$ for $i > r$)

- $u_1, \ldots u_r$ are orthonormal basis for $\mathrm{range}(A)$
- $v_1, \ldots v_r$ are orthonormal basis for $\mathcal{N}(A)^\perp$

Interpretations

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$



linear mapping $y = Ax$ can be decomposed as

- compute coefficients of x along input directions $v_1, \ldots, v_r$
- scale coefficients by σ_i
- reconstitute along output directions $u_1, \ldots, u_r$

difference with eigenvalue decomposition for symmetric A: input and output directions are *different*

- v_1 is most sensitive (highest gain) input direction
- u_1 is highest gain output direction
- $Av_1 = \sigma_1 u_1$

SVD gives clearer picture of gain as function of input/output directions

example: consider $A \in \mathbf{R}^{4\times 4}$ with $\Sigma = \mathbf{diag}(10,\ 7,\ 0.1,\ 0.05)$

- input components along directions v_1 and v_2 are amplified (by about 10) and come out mostly along plane spanned by u_1, u_2
- input components along directions v_3 and v_4 are attenuated (by about 10)
- $\|Ax\|/\|x\|$ can range between 10 and 0.05
- A is nonsingular
- for some applications you might say A is *effectively* rank 2

example: $A \in \mathbf{R}^{2\times 2}$, with $\sigma_1 = 1$, $\sigma_2 = 0.5$

- resolve x along $v_1,\ v_2$: $v_1^T x = 0.5$, $v_2^T x = 0.6$, $i.e.$, $x = 0.5 v_1 + 0.6 v_2$
- now form $Ax = (v_1^T x)\sigma_1 u_1 + (v_2^T x)\sigma_2 u_2 = (0.5)(1)u_1 + (0.6)(0.5)u_2$