Lecture 13
Linear dynamical systems with inputs & outputs

• inputs & outputs: interpretations
• transfer matrix
• impulse and step matrices
• examples
Recall continuous-time time-invariant LDS has form

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]

- \( Ax \) is called the drift term (of \( \dot{x} \))
- \( Bu \) is called the input term (of \( \dot{x} \))
Interpretations

write \( \dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m \), where \( B = [b_1 \cdots b_m] \)

- state derivative is sum of autonomous term \((Ax)\) and one term per input \((b_i u_i)\)
- each input \( u_i \) gives another degree of freedom for \( \dot{x} \) (assuming columns of \( B \) independent)

write \( \dot{x} = Ax + Bu \) as \( \dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u \), where \( \tilde{a}_i^T, \tilde{b}_i^T \) are the rows of \( A, B \)

- \( i \)th state derivative is linear function of state \( x \) and input \( u \)
Block diagram

\[ u(t) \xrightarrow{B} \dot{x}(t) \xrightarrow{1/s} x(t) \xrightarrow{C} y(t) \]

- \( A_{ij} \) is gain factor from state \( x_j \) into integrator \( i \)
- \( B_{ij} \) is gain factor from input \( u_j \) into integrator \( i \)
- \( C_{ij} \) is gain factor from state \( x_j \) into output \( y_i \)
- \( D_{ij} \) is gain factor from input \( u_j \) into output \( y_i \)
interesting when there is structure, \( e.g., \) with \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}: \)

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

- \( x_2 \) is not affected by input \( u \), \( i.e., \) \( x_2 \) propagates autonomously
- \( x_2 \) affects \( y \) directly and through \( x_1 \)
Transfer matrix

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

• $e^{tA}x(0)$ is the unforced or autonomous response

• $e^{tA}B$ is called the input-to-state impulse matrix

• $(sI - A)^{-1}B$ is called the input-to-state transfer matrix or transfer function
with \( y = Cx + Du \) we have:

\[
Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)
\]

so

\[
y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A} Bu(\tau) \, d\tau + Du(t)
\]

- output term \( Ce^{tA}x(0) \) due to initial condition
- \( H(s) = C(sI - A)^{-1}B + D \) is called the transfer function or transfer matrix
- \( h(t) = Ce^{tA}B + D\delta(t) \) is called the impulse matrix or impulse response (\( \delta \) is the Dirac delta function)
with zero initial condition we have:

\[ Y(s) = H(s)U(s), \quad y = h \ast u \]

where \( \ast \) is convolution (of matrix valued functions)

interpretation:

- \( H_{ij} \) is transfer function from input \( u_j \) to output \( y_i \)
Impulse matrix

impulse matrix \( h(t) = C e^{tA} B + D \delta(t) \)

with \( x(0) = 0, \ y = h \ast u \), i.e.,

\[
y_i(t) = \sum_{j=1}^{m} \int_{0}^{t} h_{ij}(t - \tau) u_j(\tau) \, d\tau
\]

interpretations:

- \( h_{ij}(t) \) is impulse response from \( j \)th input to \( i \)th output
- \( h_{ij}(t) \) gives \( y_i \) when \( u(t) = e_j \delta \)
- \( h_{ij}(\tau) \) shows how dependent output \( i \) is, on what input \( j \) was, \( \tau \) seconds ago
- \( i \) indexes output; \( j \) indexes input; \( \tau \) indexes time lag
Step matrix

the *step matrix* or *step response matrix* is given by

\[ s(t) = \int_0^t h(\tau) \, d\tau \]

**interpretations:**

- \( s_{ij}(t) \) is step response from \( j \)th input to \( i \)th output
- \( s_{ij}(t) \) gives \( y_i \) when \( u = e_j \) for \( t \geq 0 \)

for invertible \( A \), we have

\[ s(t) = CA^{-1} \left( e^{tA} - I \right) B + D \]
Example 1

- unit masses, springs, dampers
- $u_1$ is tension between 1st & 2nd masses
- $u_2$ is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$
system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2 \\
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}
\]

eigenvalues of $A$ are

\[-1.71 \pm j0.71, \quad -1.00 \pm j1.00, \quad -0.29 \pm j0.71\]
impulse matrix:

roughly speaking:

- impulse at $u_1$ affects third mass less than other two
- impulse at $u_2$ affects first mass later than other two
Example 2

interconnect circuit:

- $u(t) \in \mathbb{R}$ is input (drive) voltage
- $x_i$ is voltage across $C_i$
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node
system is

\[
\dot{x} = \begin{bmatrix}
-3 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -2 & 1 \\
0 & 0 & 1 & -1
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} u, \quad y = x
\]

eigenvalues of \( A \) are

\[-0.17, \quad -0.66, \quad -2.21, \quad -3.96\]
step response matrix \( s(t) \in \mathbb{R}^{4 \times 1} \):

- shortest delay to \( x_1 \); longest delay to \( x_4 \)
- delays \( \approx 10 \), consistent with slowest (i.e., dominant) eigenvalue \(-0.17\)
DC or static gain matrix

• transfer matrix at \( s = 0 \) is \( H(0) = -CA^{-1}B + D \in \mathbb{R}^{m \times p} \)

• DC transfer matrix describes system under static conditions, \( i.e., x, u, y \) constant:
\[
0 = \dot{x} = Ax + Bu, \quad y = Cx + Du
\]
eliminate \( x \) to get \( y = H(0)u \)

• if system is stable,
\[
H(0) = \int_{0}^{\infty} h(t) \, dt = \lim_{t \to \infty} s(t)
\]
(recall: \( H(s) = \int_{0}^{\infty} e^{-st}h(t) \, dt, \quad s(t) = \int_{0}^{t} h(\tau) \, d\tau \))

if \( u(t) \to u_{\infty} \in \mathbb{R}^{m} \), then \( y(t) \to y_{\infty} \in \mathbb{R}^{p} \) where \( y_{\infty} = H(0)u_{\infty} \)
DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)
Discretization with piecewise constant inputs

Linear system \( \dot{x} = Ax + Bu, \ y = Cx + Du \)

suppose \( u_d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m \) is a sequence, and

\[ u(t) = u_d(k) \quad \text{for} \quad kh \leq t < (k + 1)h, \ k = 0, 1, \ldots \]

define sequences

\[ x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \ldots \]

• \( h > 0 \) is called the sample interval (for \( x \) and \( y \)) or update interval (for \( u \))

• \( u \) is piecewise constant (called zero-order-hold)

• \( x_d, y_d \) are sampled versions of \( x, y \)
\begin{align*}
x_d(k + 1) &= x((k + 1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A}Bu((k + 1)h - \tau) \, d\tau \\
&= e^{hA}x_d(k) + \left( \int_0^h e^{\tau A} \, d\tau \right) B \, u_d(k)
\end{align*}

\( x_d, \ u_d, \) and \( y_d \) satisfy discrete-time LDS equations

\[ x_d(k + 1) = A_dx_d(k) + B_du_d(k), \quad y_d(k) = C_dx_d(k) + D_du_d(k) \]

where

\[ A_d = e^{hA}, \quad B_d = \left( \int_0^h e^{\tau A} \, d\tau \right) B, \quad C_d = C, \quad D_d = D \]
called *discretized system*

if $A$ is invertible, we can express integral as

$$
\int_0^h e^{\tau A} \, d\tau = A^{-1} \left( e^{hA} - I \right)
$$

**stability:** if eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of $A_d$ are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$
\Re \lambda_i < 0 \iff |e^{h\lambda_i}| < 1
$$

for $h > 0$
extensions/variations:

- offsets: updates for $u$ and sampling of $x$, $y$ are offset in time

- multirate: $u_i$ updated, $y_i$ sampled at different intervals
  (usually integer multiples of a common interval $h$)

both very common in practice
Dual system

the dual system associated with system

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du
\]

is given by

\[
\dot{z} = A^T z + C^T v, \quad w = B^T z + D^T v
\]

- all matrices are transposed
- role of $B$ and $C$ are swapped

transfer function of dual system:

\[
(B^T)(sI - A^T)^{-1}(C^T) + D^T = H(s)^T
\]

where $H(s) = C(sI - A)^{-1}B + D$
(for SISO case, TF of dual is same as original)
eigenvalues (hence stability properties) are the same
Dual via block diagram

in terms of block diagrams, dual is formed by:

- transpose all matrices
- swap inputs and outputs on all boxes
- reverse directions of signal flow arrows
- swap solder joints and summing junctions
original system:

\[
\begin{align*}
u(t) &\rightarrow B & \rightarrow 1/s & \rightarrow x(t) &\rightarrow C &\rightarrow y(t) \\
A &\rightarrow D
\end{align*}
\]

dual system:

\[
\begin{align*}
w(t) &\rightarrow B^T & \rightarrow 1/s & \rightarrow z(t) &\rightarrow C^T &\rightarrow v(t) \\
A^T &\rightarrow D^T
\end{align*}
\]
Causality

interpretation of

\[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \]

\[ y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) \, d\tau + Du(t) \]

for \( t \geq 0 \):

*current* state \((x(t))\) and output \((y(t))\) depend on *past* input \((u(\tau)\) for \( \tau \leq t \))

*i.e.*, mapping from input to state and output is *causal* (with fixed *initial* state)
now consider fixed final state $x(T)$: for $t \leq T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau) \, d\tau,$$

i.e., current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal
Idea of state

$x(t)$ is called state of system at time $t$ since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs
Change of coordinates

start with LDS $\dot{x} = Ax + Bu, \ y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since $u, \ y$ aren’t affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$
Standard forms for LDS

can change coordinates to put $A$ in various forms (diagonal, real modal, Jordan . . .)

e.g., to put LDS in diagonal form, find $T$ s.t.

$$T^{-1}AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

so

$$\dot{x}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \quad y = \sum_{i=1}^n \tilde{c}_i \tilde{x}_i$$
(here we assume $D = 0$)
Discrete-time systems

discrete-time LDS:

\[ x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

- only difference w/cts-time: \( z \) instead of \( s \)
- interpretation of \( z^{-1} \) block:
  - unit delayor (shifts sequence back in time one epoch)
  - latch (plus small delay to avoid race condition)
we have:

\[ x(1) = Ax(0) + Bu(0), \]

\[ x(2) = Ax(1) + Bu(1) \]
\[ = A^2x(0) + ABu(0) + Bu(1), \]

and in general, for \( t \in \mathbb{Z}_+ \),

\[ x(t) = A^tx(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)}Bu(\tau) \]

hence

\[ y(t) = CA^tx(0) + h \ast u \]
where $\ast$ is discrete-time convolution and

$$h(t) = \begin{cases} 
D, & t = 0 \\
CA^{t-1}B, & t > 0 
\end{cases}$$

is the impulse response
\textbf{\(\mathcal{Z}\)-transform}

suppose \(w \in \mathbb{R}^{p \times q}\) is a sequence (discrete-time signal), \(i.e.,\)

\[w : \mathbb{Z}_+ \rightarrow \mathbb{R}^{p \times q}\]

recall \(\mathcal{Z}\)-transform \(W = \mathcal{Z}(w)\):

\[W(z) = \sum_{t=0}^{\infty} z^{-t}w(t)\]

where \(W : D \subseteq \mathbb{C} \rightarrow \mathbb{C}^{p \times q}\) \((D\) is domain of \(W)\)

time-advanced or shifted signal \(v\):

\[v(t) = w(t + 1), \quad t = 0, 1, \ldots\]
\( \mathcal{Z} \)-transform of time-advanced signal:

\[
V(z) = \sum_{t=0}^{\infty} z^{-t} w(t + 1)
\]

\[
= z \sum_{t=1}^{\infty} z^{-t} w(t)
\]

\[
= zW(z) - zw(0)
\]
Discrete-time transfer function

take $\mathcal{Z}$-transform of system equations

$$
x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
$$

yields

$$
zX(z) - zx(0) = AX(z) + BU(z), \quad Y(z) = CX(z) + DU(z)
$$

solve for $X(z)$ to get

$$
X(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}BU(z)
$$

(note extra $z$ in first term!)
hence

\[ Y(z) = H(z)U(z) + C(zI - A)^{-1}zx(0) \]

where \( H(z) = C(zI - A)^{-1}B + D \) is the \textit{discrete-time transfer function}

note power series expansion of resolvent:

\[
(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots
\]