

# Lecture 2

## Linear functions and examples

- linear equations and functions
- engineering examples
- interpretations

# Linear equations

consider system of linear equations

$$\begin{aligned}y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\&\vdots \\y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{aligned}$$

can be written in matrix form as  $y = Ax$ , where

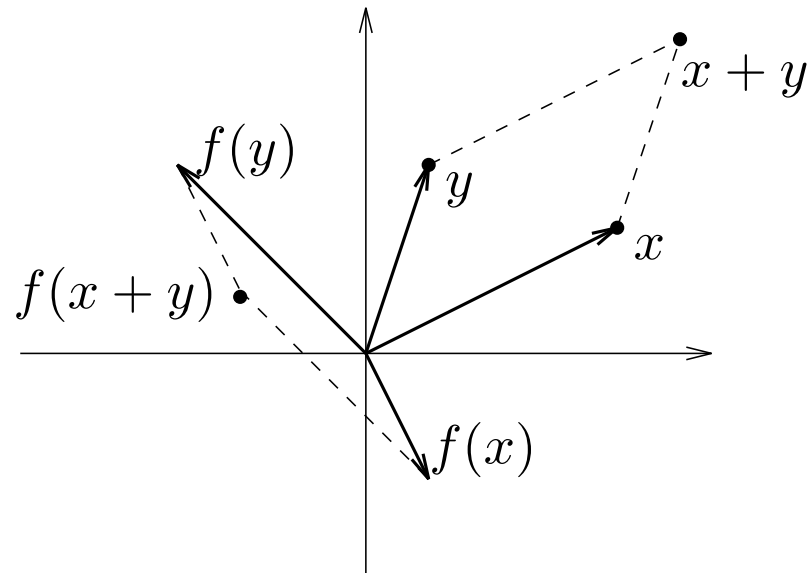
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

# Linear functions

a function  $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$  is *linear* if

- $f(x + y) = f(x) + f(y), \forall x, y \in \mathbf{R}^n$
- $f(\alpha x) = \alpha f(x), \forall x \in \mathbf{R}^n \forall \alpha \in \mathbf{R}$

*i.e.*, *superposition* holds



# Matrix multiplication function

- consider function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  given by  $f(x) = Ax$ , where  $A \in \mathbf{R}^{m \times n}$
- matrix multiplication function  $f$  is linear
- **converse** is true: **any** linear function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  can be written as  $f(x) = Ax$  for some  $A \in \mathbf{R}^{m \times n}$
- representation via matrix multiplication is unique: for any linear function  $f$  there is only one matrix  $A$  for which  $f(x) = Ax$  for all  $x$
- $y = Ax$  is a concrete representation of a generic linear function

## Interpretations of $y = Ax$

- $y$  is measurement or observation;  $x$  is unknown to be determined
- $x$  is 'input' or 'action';  $y$  is 'output' or 'result'
- $y = Ax$  defines a function or transformation that maps  $x \in \mathbf{R}^n$  into  $y \in \mathbf{R}^m$

## Interpretation of $a_{ij}$

$$y_i = \sum_{j=1}^n a_{ij} x_j$$

$a_{ij}$  is *gain factor* from  $j$ th input ( $x_j$ ) to  $i$ th output ( $y_i$ )

thus, *e.g.*,

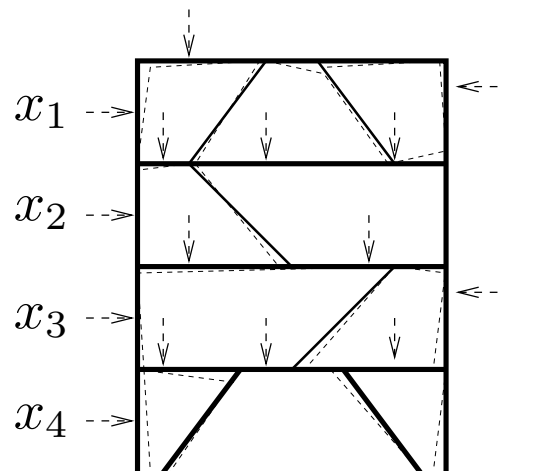
- $i$ th *row* of  $A$  concerns  $i$ th *output*
- $j$ th *column* of  $A$  concerns  $j$ th *input*
- $a_{27} = 0$  means 2nd output ( $y_2$ ) doesn't depend on 7th input ( $x_7$ )
- $|a_{31}| \gg |a_{3j}|$  for  $j \neq 1$  means  $y_3$  depends mainly on  $x_1$

- $|a_{52}| \gg |a_{i2}|$  for  $i \neq 5$  means  $x_2$  affects mainly  $y_5$
- $A$  is lower triangular, *i.e.*,  $a_{ij} = 0$  for  $i < j$ , means  $y_i$  only depends on  $x_1, \dots, x_i$
- $A$  is diagonal, *i.e.*,  $a_{ij} = 0$  for  $i \neq j$ , means  $i$ th output depends only on  $i$ th input

more generally, **sparsity pattern** of  $A$ , *i.e.*, list of zero/nonzero entries of  $A$ , shows which  $x_j$  affect which  $y_i$

# Linear elastic structure

- $x_j$  is external force applied at some node, in some fixed direction
- $y_i$  is (small) deflection of some node, in some fixed direction

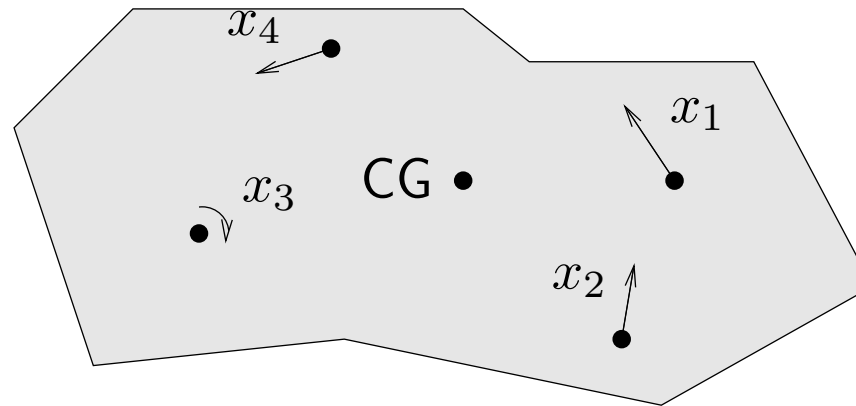


(provided  $x$ ,  $y$  are small) we have  $y \approx Ax$

- $A$  is called the *compliance matrix*
- $a_{ij}$  gives deflection  $i$  per unit force at  $j$  (in m/N)



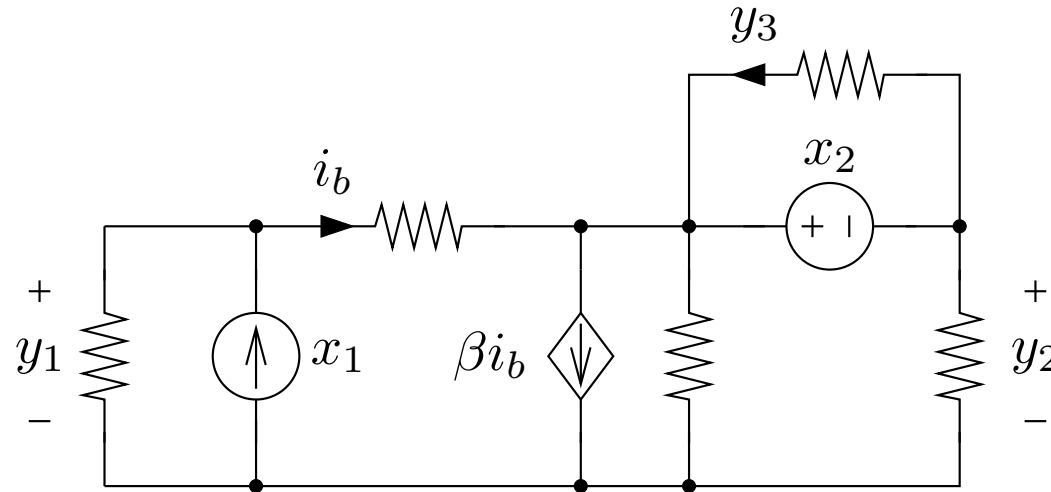
# Total force/torque on rigid body



- $x_j$  is external force/torque applied at some point/direction/axis
- $y \in \mathbf{R}^6$  is resulting total force & torque on body  
( $y_1, y_2, y_3$  are  $x$ -,  $y$ -,  $z$ - components of total force,  
 $y_4, y_5, y_6$  are  $x$ -,  $y$ -,  $z$ - components of total torque)
- we have  $y = Ax$
- $A$  depends on geometry  
(of applied forces and torques with respect to center of gravity CG)
- $j$ th column gives resulting force & torque for unit force/torque  $j$

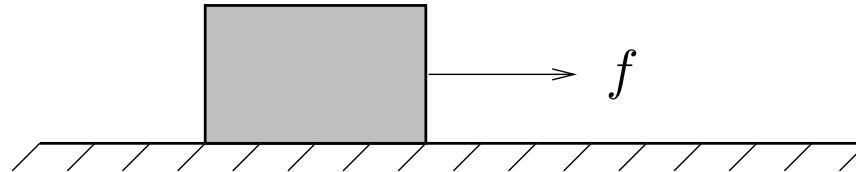
# Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



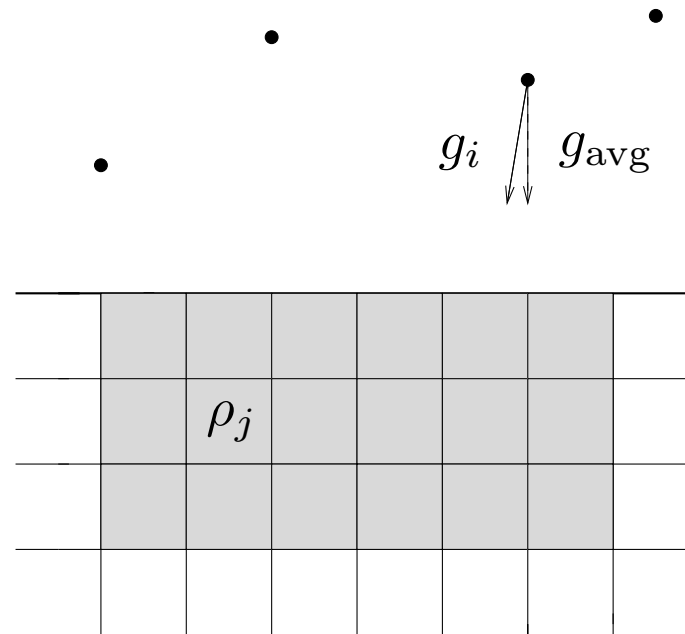
- $x_j$  is value of independent source  $j$
- $y_i$  is some circuit variable (voltage, current)
- we have  $y = Ax$
- if  $x_j$  are currents and  $y_i$  are voltages,  $A$  is called the *impedance* or *resistance* matrix

# Final position/velocity of mass due to applied forces



- unit mass, zero position/velocity at  $t = 0$ , subject to force  $f(t)$  for  $0 \leq t \leq n$
- $f(t) = x_j$  for  $j - 1 \leq t < j$ ,  $j = 1, \dots, n$   
( $x$  is the sequence of applied forces, constant in each interval)
- $y_1, y_2$  are final position and velocity (*i.e.*, at  $t = n$ )
- we have  $y = Ax$
- $a_{1j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final position
- $a_{2j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final velocity

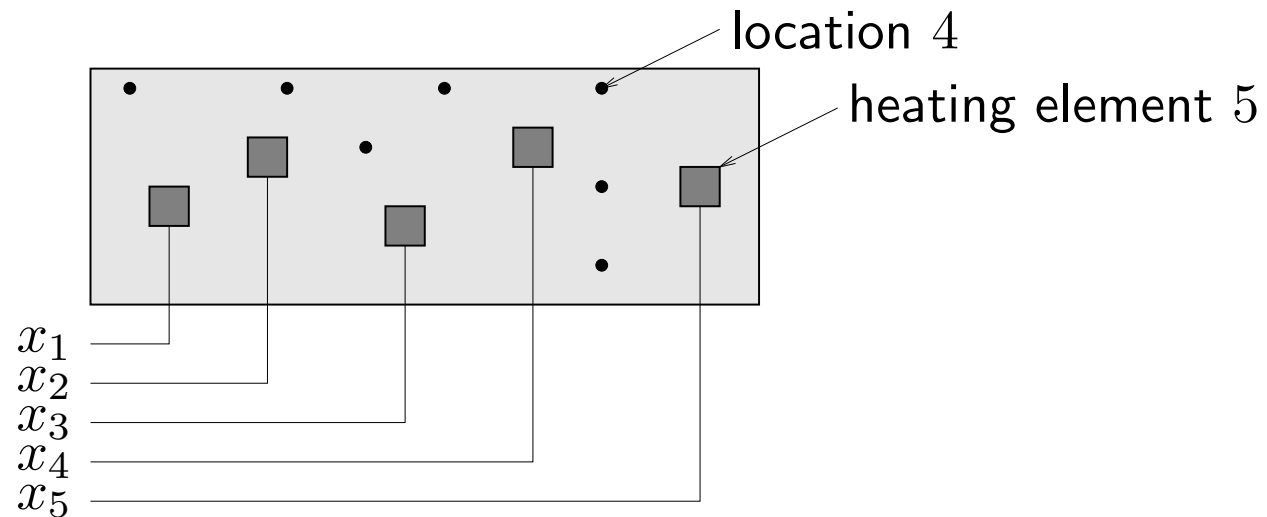
# Gravimeter prospecting



- $x_j = \rho_j - \rho_{\text{avg}}$  is (excess) mass density of earth in voxel  $j$ ;
- $y_i$  is measured *gravity anomaly* at location  $i$ , *i.e.*, some component (typically vertical) of  $g_i - g_{\text{avg}}$
- $y = Ax$

- $A$  comes from physics and geometry
- $j$ th column of  $A$  shows sensor readings caused by unit density anomaly at voxel  $j$
- $i$ th row of  $A$  shows sensitivity pattern of sensor  $i$

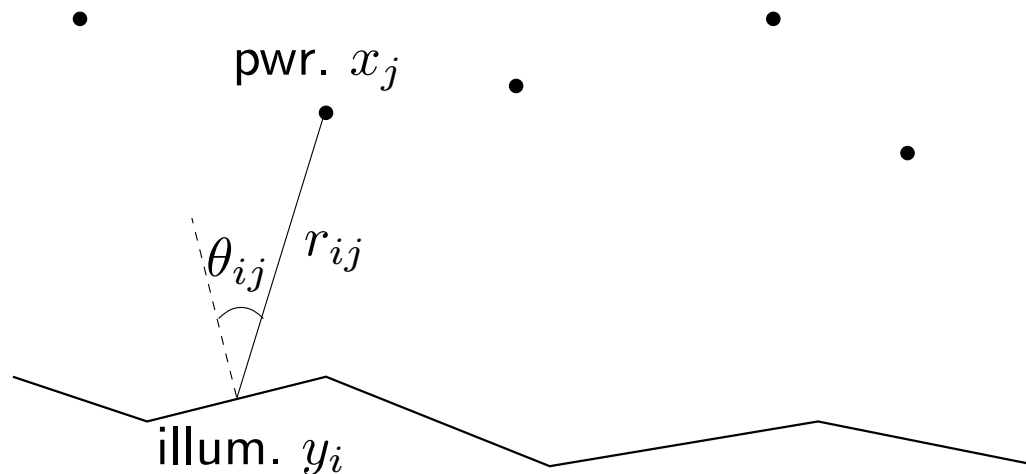
# Thermal system



- $x_j$  is power of  $j$ th heating element or heat source
- $y_i$  is change in steady-state temperature at location  $i$
- thermal transport via conduction
- $y = Ax$

- $a_{ij}$  gives influence of heater  $j$  at location  $i$  (in  $^{\circ}\text{C}/\text{W}$ )
- $j$ th column of  $A$  gives pattern of steady-state temperature rise due to 1W at heater  $j$
- $i$ th row shows how heaters affect location  $i$

# Illumination with multiple lamps



- $n$  lamps illuminating  $m$  (small, flat) patches, no shadows
- $x_j$  is power of  $j$ th lamp;  $y_i$  is illumination level of patch  $i$
- $y = Ax$ , where  $a_{ij} = r_{ij}^{-2} \max\{\cos \theta_{ij}, 0\}$   
( $\cos \theta_{ij} < 0$  means patch  $i$  is shaded from lamp  $j$ )
- $j$ th column of  $A$  shows illumination pattern from lamp  $j$



# Signal and interference power in wireless system

- $n$  transmitter/receiver pairs
- transmitter  $j$  transmits to receiver  $j$  (and, inadvertently, to the other receivers)
- $p_j$  is power of  $j$ th transmitter
- $s_i$  is received signal power of  $i$ th receiver
- $z_i$  is received interference power of  $i$ th receiver
- $G_{ij}$  is path gain from transmitter  $j$  to receiver  $i$
- we have  $s = Ap$ ,  $z = Bp$ , where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \quad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- $A$  is diagonal;  $B$  has zero diagonal (ideally,  $A$  is 'large',  $B$  is 'small')

# Cost of production

production *inputs* (materials, parts, labor, . . . ) are combined to make a number of *products*

- $x_j$  is price per unit of production input  $j$
- $a_{ij}$  is units of production input  $j$  required to manufacture one unit of product  $i$
- $y_i$  is production cost per unit of product  $i$
- we have  $y = Ax$
- $i$ th row of  $A$  is *bill of materials* for unit of product  $i$

## production inputs needed

- $q_i$  is quantity of product  $i$  to be produced
- $r_j$  is total quantity of production input  $j$  needed
- we have  $r = A^T q$

total production cost is

$$r^T x = (A^T q)^T x = q^T Ax$$

# Network traffic and flows

- $n$  flows with rates  $f_1, \dots, f_n$  pass from their source nodes to their destination nodes over fixed routes in a network
- $t_i$ , traffic on link  $i$ , is sum of rates of flows passing through it
- flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- traffic and flow rates related by  $t = Af$

## link delays and flow latency

- let  $d_1, \dots, d_m$  be link delays, and  $l_1, \dots, l_n$  be latency (total travel time) of flows
- $l = A^T d$
- $f^T l = f^T A^T d = (Af)^T d = t^T d$ , total # of packets in network

# Linearization

- if  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is differentiable at  $x_0 \in \mathbf{R}^n$ , then

$$x \text{ near } x_0 \implies f(x) \text{ very near } f(x_0) + Df(x_0)(x - x_0)$$

where

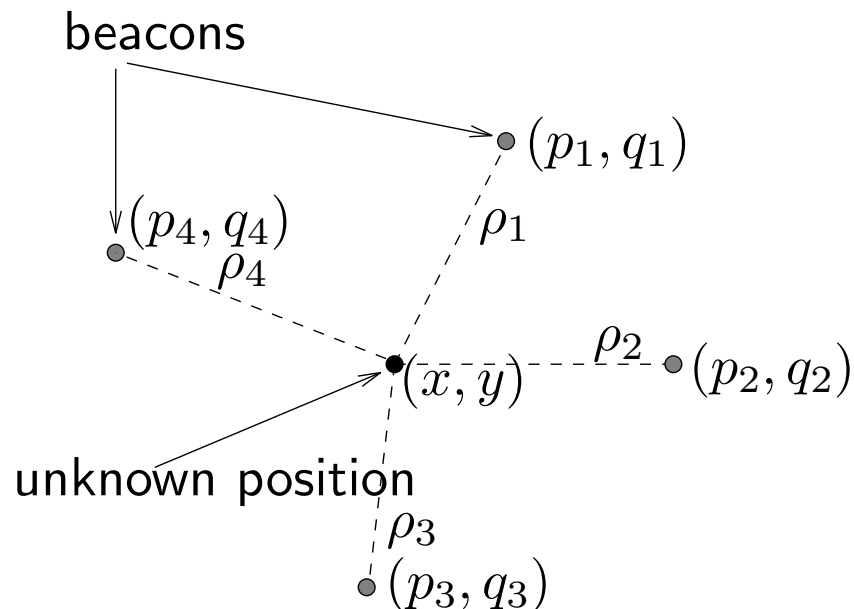
$$Df(x_0)_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{x_0}$$

is derivative (Jacobian) matrix

- with  $y = f(x)$ ,  $y_0 = f(x_0)$ , define *input deviation*  $\delta x := x - x_0$ , *output deviation*  $\delta y := y - y_0$
- then we have  $\delta y \approx Df(x_0)\delta x$
- when deviations are small, they are (approximately) related by a linear function

# Navigation by range measurement

- $(x, y)$  unknown coordinates in plane
- $(p_i, q_i)$  known coordinates of beacons for  $i = 1, 2, 3, 4$
- $\rho_i$  measured (known) distance or range from beacon  $i$



- $\rho \in \mathbf{R}^4$  is a nonlinear function of  $(x, y) \in \mathbf{R}^2$ :

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- linearize around  $(x_0, y_0)$ :  $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$ , where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- $i$ th row of  $A$  shows (approximate) change in  $i$ th range measurement for (small) shift in  $(x, y)$  from  $(x_0, y_0)$
- first column of  $A$  shows sensitivity of range measurements to (small) change in  $x$  from  $x_0$
- obvious application:  $(x_0, y_0)$  is last navigation fix;  $(x, y)$  is current position, a short time later



# Broad categories of applications

linear model or function  $y = Ax$

some broad categories of applications:

- estimation or inversion
- control or design
- mapping or transformation

(this list is not exclusive; can have combinations . . . )

# Estimation or inversion

$$y = Ax$$

- $y_i$  is  $i$ th measurement or sensor reading (which we know)
- $x_j$  is  $j$ th parameter to be estimated or determined
- $a_{ij}$  is sensitivity of  $i$ th sensor to  $j$ th parameter

sample problems:

- find  $x$ , given  $y$
- find all  $x$ 's that result in  $y$  (*i.e.*, all  $x$ 's consistent with measurements)
- if there is no  $x$  such that  $y = Ax$ , find  $x$  s.t.  $y \approx Ax$  (*i.e.*, if the sensor readings are inconsistent, find  $x$  which is almost consistent)

# Control or design

$$y = Ax$$

- $x$  is vector of design parameters or inputs (which we can choose)
- $y$  is vector of results, or outcomes
- $A$  describes how input choices affect results

sample problems:

- find  $x$  so that  $y = y_{\text{des}}$
- find all  $x$ 's that result in  $y = y_{\text{des}}$  (*i.e.*, find all designs that meet specifications)
- among  $x$ 's that satisfy  $y = y_{\text{des}}$ , find a small one (*i.e.*, find a small or efficient  $x$  that meets specifications)

# Mapping or transformation

- $x$  is mapped or transformed to  $y$  by linear function  $y = Ax$

sample problems:

- determine if there is an  $x$  that maps to a given  $y$
- (if possible) find *an*  $x$  that maps to  $y$
- find *all*  $x$ 's that map to a given  $y$
- if there is only one  $x$  that maps to  $y$ , find it (*i.e.*, decode or undo the mapping)

# Matrix multiplication as mixture of columns

write  $A \in \mathbf{R}^{m \times n}$  in terms of its columns:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

where  $a_j \in \mathbf{R}^m$

then  $y = Ax$  can be written as

$$y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

( $x_j$ 's are scalars,  $a_j$ 's are  $m$ -vectors)

- $y$  is a (linear) combination or mixture of the columns of  $A$
- coefficients of  $x$  give coefficients of mixture

an important example:  $x = e_j$ , the  $j$ th *unit vector*

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

then  $Ae_j = a_j$ , the  $j$ th column of  $A$

( $e_j$  corresponds to a pure mixture, giving only column  $j$ )

# Matrix multiplication as inner product with rows

write  $A$  in terms of its rows:

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_n^T \end{bmatrix}$$

where  $\tilde{a}_i \in \mathbf{R}^n$

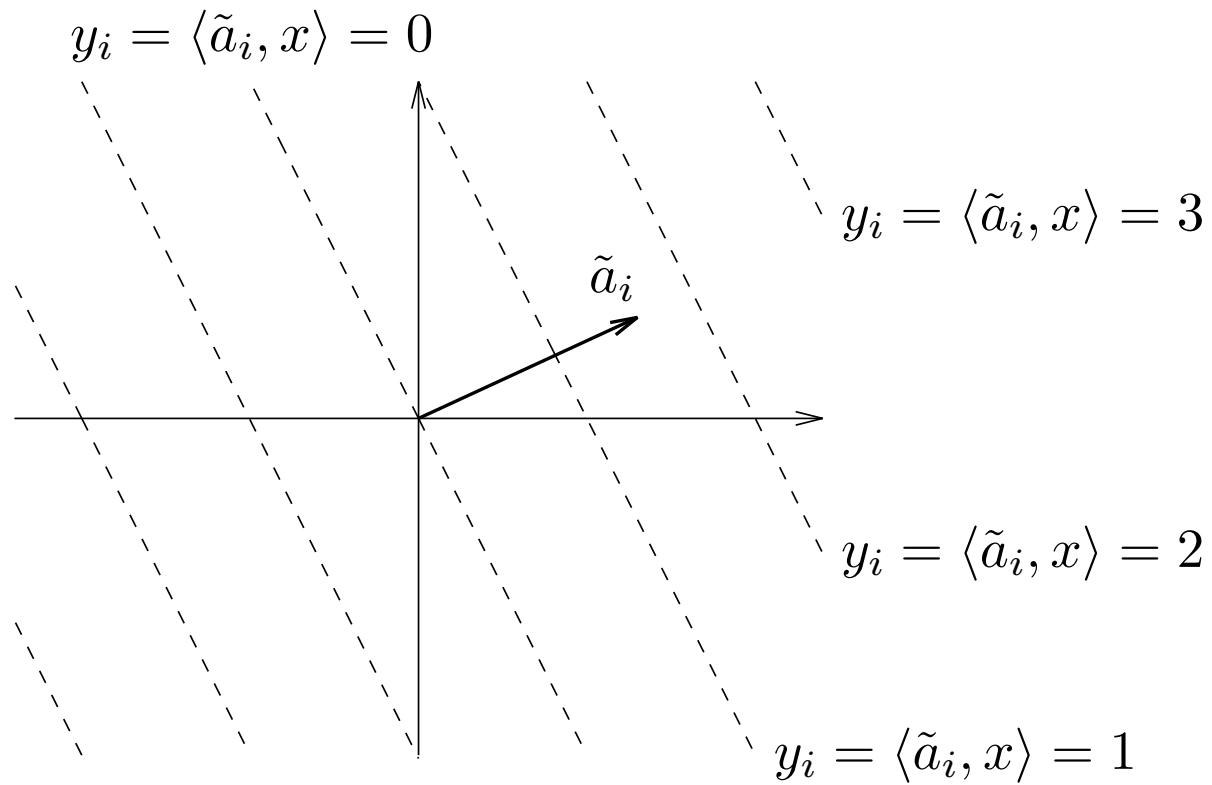
then  $y = Ax$  can be written as

$$y = \begin{bmatrix} \tilde{a}_1^T x \\ \tilde{a}_2^T x \\ \vdots \\ \tilde{a}_m^T x \end{bmatrix}$$

thus  $y_i = \langle \tilde{a}_i, x \rangle$ , *i.e.*,  $y_i$  is inner product of  $i$ th row of  $A$  with  $x$

## geometric interpretation:

$y_i = \tilde{a}_i^T x = \alpha$  is a hyperplane in  $\mathbf{R}^n$  (normal to  $\tilde{a}_i$ )





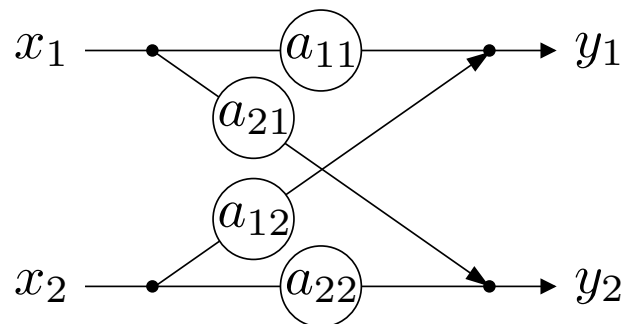
## Block diagram representation

$y = Ax$  can be represented by a *signal flow graph* or *block diagram*

*e.g.* for  $m = n = 2$ , we represent

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as



- $a_{ij}$  is the gain along the path from  $j$ th input to  $i$ th output
- (by not drawing paths with zero gain) shows sparsity structure of  $A$  (*e.g.*, diagonal, block upper triangular, arrow . . . )

**example:** block upper triangular, *i.e.*,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where  $A_{11} \in \mathbf{R}^{m_1 \times n_1}$ ,  $A_{12} \in \mathbf{R}^{m_1 \times n_2}$ ,  $A_{21} \in \mathbf{R}^{m_2 \times n_1}$ ,  $A_{22} \in \mathbf{R}^{m_2 \times n_2}$

partition  $x$  and  $y$  conformably as

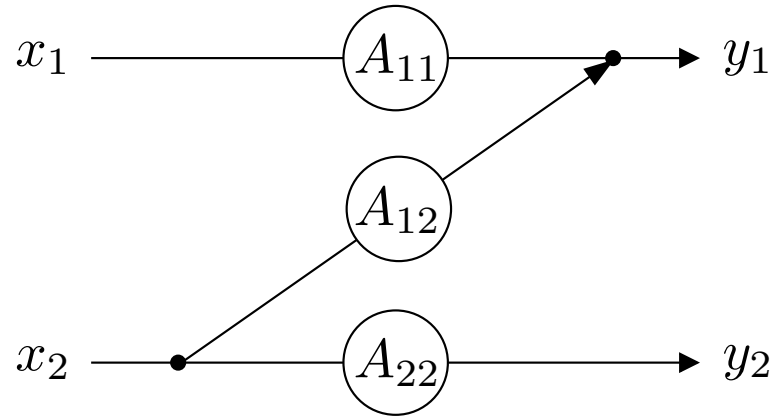
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

( $x_1 \in \mathbf{R}^{n_1}$ ,  $x_2 \in \mathbf{R}^{n_2}$ ,  $y_1 \in \mathbf{R}^{m_1}$ ,  $y_2 \in \mathbf{R}^{m_2}$ ) so

$$y_1 = A_{11}x_1 + A_{12}x_2, \quad y_2 = A_{22}x_2,$$

*i.e.*,  $y_2$  doesn't depend on  $x_1$

block diagram:



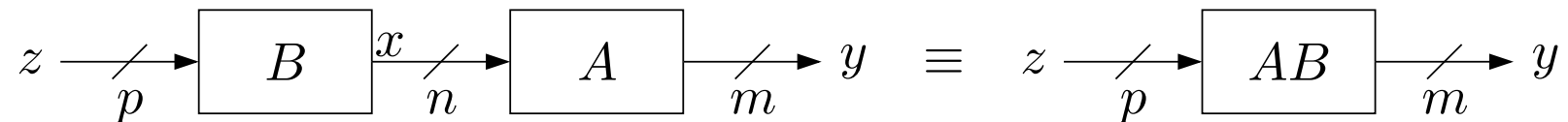
. . . no path from  $x_1$  to  $y_2$ , so  $y_2$  doesn't depend on  $x_1$

# Matrix multiplication as composition

for  $A \in \mathbf{R}^{m \times n}$  and  $B \in \mathbf{R}^{n \times p}$ ,  $C = AB \in \mathbf{R}^{m \times p}$  where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

**composition interpretation:**  $y = Cz$  represents composition of  $y = Ax$  and  $x = Bz$



(note that  $B$  is on left in block diagram)

## Column and row interpretations

can write product  $C = AB$  as

$$C = [ c_1 \cdots c_p ] = AB = [ Ab_1 \cdots Ab_p ]$$

*i.e.*,  $i$ th column of  $C$  is  $A$  acting on  $i$ th column of  $B$

similarly we can write

$$C = \begin{bmatrix} \tilde{c}_1^T \\ \vdots \\ \tilde{c}_m^T \end{bmatrix} = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$$

*i.e.*,  $i$ th row of  $C$  is  $i$ th row of  $A$  acting (on left) on  $B$

# Inner product interpretation

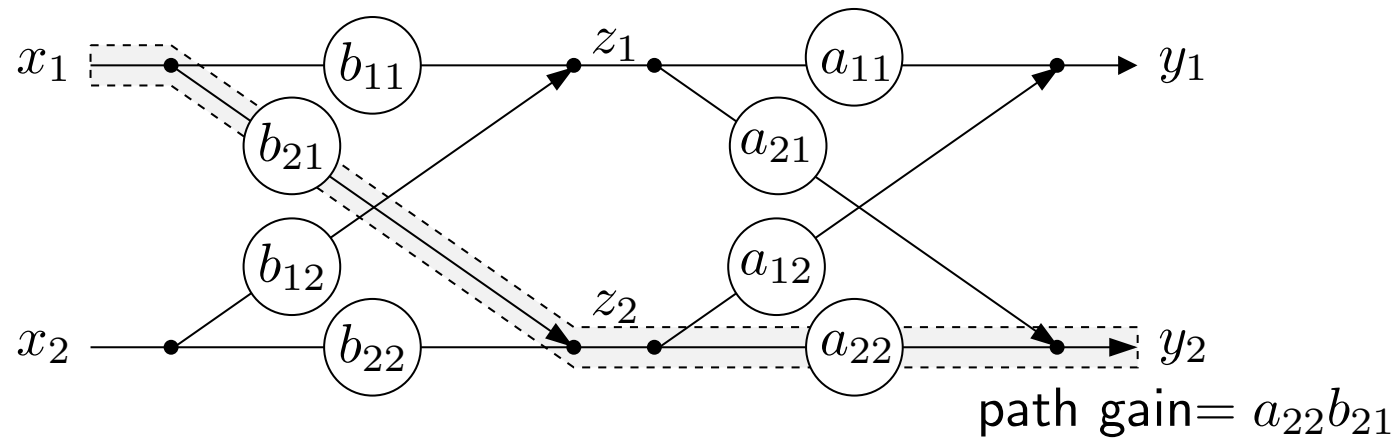
inner product interpretation:

$$c_{ij} = \tilde{a}_i^T b_j = \langle \tilde{a}_i, b_j \rangle$$

*i.e.*, entries of  $C$  are inner products of rows of  $A$  and columns of  $B$

- $c_{ij} = 0$  means  $i$ th row of  $A$  is orthogonal to  $j$ th column of  $B$
- **Gram matrix** of vectors  $f_1, \dots, f_n$  defined as  $G_{ij} = f_i^T f_j$   
(gives inner product of each vector with the others)
- $G = [f_1 \ \cdots \ f_n]^T [f_1 \ \cdots \ f_n]$

# Matrix multiplication interpretation via paths



- $a_{ik}b_{kj}$  is gain of path from input  $j$  to output  $i$  via  $k$
- $c_{ij}$  is sum of gains over *all* paths from input  $j$  to output  $i$