Lecture 18
Controllability and state transfer

• state transfer

• reachable set, controllability matrix

• minimum norm inputs

• infinite-horizon minimum norm transfer
State transfer

consider $\dot{x} = Ax + Bu$ (or $x(t + 1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$

we say input $u : [t_i, t_f] \to \mathbb{R}^m$ steers or transfers state from $x(t_i)$ to $x(t_f)$ (over time interval $[t_i, t_f]$)

(subscripts stand for initial and final)

questions:

- where can $x(t_i)$ be transfered to at $t = t_f$?

- how quickly can $x(t_i)$ be transfered to some $x_{\text{target}}$?

- how do we find a $u$ that transfers $x(t_i)$ to $x(t_f)$?

- how do we find a ‘small’ or ‘efficient’ $u$ that transfers $x(t_i)$ to $x(t_f)$?
Reachability

consider state transfer from $x(0) = 0$ to $x(t)$

we say $x(t)$ is reachable (in $t$ seconds or epochs)

we define $\mathcal{R}_t \subseteq \mathbb{R}^n$ as the set of points reachable in $t$ seconds or epochs

for CT system $\dot{x} = Ax + Bu$,

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) \, d\tau \mid u : [0, t] \to \mathbb{R}^m \right\}$$

and for DT system $x(t+1) = Ax(t) + Bu(t)$,

$$\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \mid u(t) \in \mathbb{R}^m \right\}$$
- $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$

- $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$

  (i.e., can reach more points given more time)

We define the *reachable set* $\mathcal{R}$ as the set of points reachable for some $t$:

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$
Reachability for discrete-time LDS

DT system \( x(t + 1) = Ax(t) + Bu(t), \ x(t) \in \mathbb{R}^n \)

\[
x(t) = C_t \begin{bmatrix} u(t - 1) \\ \vdots \\ u(0) \end{bmatrix}
\]

where \( C_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \)

so reachable set at \( t \) is \( \mathcal{R}_t = \text{range}(C_t) \)

by C-H theorem, we can express each \( A^k \) for \( k \geq n \) as linear combination of \( A^0, \ldots, A^{n-1} \)

hence for \( t \geq n \), \( \text{range}(C_t) = \text{range}(C_n) \)
thus we have

\[ \mathcal{R}_t = \begin{cases} \text{range}(C_t) & t < n \\ \text{range}(C) & t \geq n \end{cases} \]

where \( C = C_n \) is called the *controllability matrix*.

- any state that can be reached can be reached by \( t = n \)
- the reachable set is \( \mathcal{R} = \text{range}(C) \)
Controllable system

system is called \textit{reachable} or \textit{controllable} if all states are reachable (\(i.e., \mathcal{R} = \mathbb{R}^n\))

system is reachable if and only if \textbf{Rank}(\(C\)) = \(n\)

\textbf{example:} \(x(t + 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)\)

controllability matrix is \(C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\)

hence system is not controllable; reachable set is

\[ \mathcal{R} = \text{range}(C) = \{ x \mid x_1 = x_2 \} \]
General state transfer

with \( t_f > t_i \),

\[
x(t_f) = A^{t_f-t_i}x(t_i) + C_{t_f-t_i} \begin{bmatrix} u(t_f-1) \\ \vdots \\ u(t_i) \end{bmatrix}
\]

hence can transfer \( x(t_i) \) to \( x(t_f) = x_{\text{des}} \)

\[\Leftrightarrow \quad x_{\text{des}} - A^{t_f-t_i}x(t_i) \in \mathcal{R}_{t_f-t_i}\]

• general state transfer reduces to reachability problem
• if system is controllable any state transfer can be achieved in \( \leq n \) steps
• important special case: driving state to zero (sometimes called regulating or controlling state)
Least-norm input for reachability

assume system is reachable, $\text{Rank}(C_t) = n$

to steer $x(0) = 0$ to $x(t) = x_{\text{des}}$, inputs $u(0), \ldots, u(t - 1)$ must satisfy

$$x_{\text{des}} = C_t \begin{bmatrix} u(t - 1) \\ \vdots \\ u(0) \end{bmatrix}$$

among all $u$ that steer $x(0) = 0$ to $x(t) = x_{\text{des}}$, the one that minimizes

$$\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$$
is given by
\[
\begin{bmatrix}
    u_{\ln}(t - 1) \\
    \vdots \\
    u_{\ln}(0)
\end{bmatrix} = C_t^T (C_t C_t^T)^{-1} x_{\text{des}}
\]

\( u_{\ln} \) is called \textit{least-norm} or \textit{minimum energy} input that effects state transfer can express as

\[
u_{\ln}(\tau) = B^T (A^T)^{(t-1-\tau)} \left( \sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}},
\]

for \( \tau = 0, \ldots, t - 1 \)
\( \mathcal{E}_{\text{min}} \), the minimum value of \( \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \) required to reach \( x(t) = x_{\text{des}} \), is sometimes called *minimum energy* required to reach \( x(t) = x_{\text{des}} \)

\[
\mathcal{E}_{\text{min}} = \sum_{\tau=0}^{t-1} \|u_{\ln}(\tau)\|^2 \\
= (C_t^T (C_tC_t)^{-1} \cdot x_{\text{des}})^T C_t^T (C_tC_t)^{-1} \cdot x_{\text{des}} \\
= x_{\text{des}}^T (C_tC_t)^{-1} \cdot x_{\text{des}} \\
= x_{\text{des}}^T \left( \sum_{\tau=0}^{t-1} A^T B B^T (A^T)^\tau \right)^{-1} \cdot x_{\text{des}}
\]
• $\mathcal{E}_{\text{min}}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (i.e., how large a $u$ is required)

• $\mathcal{E}_{\text{min}}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of $x_{\text{des}}, t$)

• ellipsoid \( \{ z \mid \mathcal{E}_{\text{min}}(z, t) \leq 1 \} \) shows points in state space reachable at $t$ with one unit of energy
  
  (shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)
\( E_{\min} \) as function of \( t \):

if \( t \geq s \) then

\[
\sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \geq \sum_{\tau=0}^{s-1} A^\tau B B^T (A^T)^\tau
\]

hence

\[
\left( \sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1} \leq \left( \sum_{\tau=0}^{s-1} A^\tau B B^T (A^T)^\tau \right)^{-1}
\]

so \( E_{\min}(x_{\text{des}}, t) \leq E_{\min}(x_{\text{des}}, s) \)

i.e.: takes less energy to get somewhere more leisurely
**example:** \( x(t + 1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \)

\[ \mathcal{E}_{\text{min}}(z, t) \text{ for } z = [1 \ 1]^T: \]
ellipsoids $\mathcal{E}_{\text{min}} \leq 1$ for $t = 3$ and $t = 10$: 

\begin{align*}
\mathcal{E}_{\text{min}}(x, 3) &\leq 1 \\
\mathcal{E}_{\text{min}}(x, 10) &\leq 1
\end{align*}
Minimum energy over infinite horizon

the matrix

\[
P = \lim_{t \to \infty} \left( \sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1}
\]

always exists, and gives the minimum energy required to reach a point \( x_{\text{des}} \) (with no limit on \( t \)):

\[
\min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \mid x(0) = 0, \ x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}
\]

if \( A \) is stable, \( P > 0 \) (i.e., can’t get anywhere for free)

if \( A \) is not stable, then \( P \) can have nonzero nullspace
• $Pz = 0, z \neq 0$ means can get to $z$ using $u$’s with energy as small as you like

($u$ just gives a little kick to the state; the instability carries it out to $z$ efficiently)

• basis of highly maneuverable, unstable aircraft
Continuous-time reachability

consider now $\dot{x} = Ax + Bu$ with $x(t) \in \mathbb{R}^n$

reachable set at time $t$ is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \mid u : [0, t] \to \mathbb{R}^m \right\}$$

**fact:** for $t > 0$, $\mathcal{R}_t = \mathcal{R} = \text{range}(C)$, where

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is the controllability matrix of $(A, B)$

- same $\mathcal{R}$ as discrete-time system
- for continuous-time system, any reachable point can be reached as fast as you like (with large enough $u$)
first let’s show for any $u$ (and $x(0) = 0$) we have $x(t) \in \text{range}(C)$

write $e^{tA}$ as power series:

$$e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots$$

by C-H, express $A^n, A^{n+1}, \ldots$ in terms of $A^0, \ldots, A^{n-1}$ and collect powers of $A$:

$$e^{tA} = \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha_{n-1}(t)A^{n-1}$$

therefore

$$x(t) = \int_0^t e^{\tau A} Bu(t - \tau) \, d\tau$$

$$= \int_0^t \left( \sum_{i=0}^{n-1} \alpha_i(\tau) A^i \right) Bu(t - \tau) \, d\tau$$
\[
\begin{align*}
&= \sum_{i=0}^{n-1} A^i B \int_{0}^{t} \alpha_i(\tau) u(t - \tau) \, d\tau \\
&= Cz
\end{align*}
\]

where \( z_i = \int_{0}^{t} \alpha_i(\tau) u(t - \tau) \, d\tau \)

hence, \( x(t) \) is always in \( \text{range}(C) \)

need to show converse: every point in \( \text{range}(C) \) can be reached
Impulsive inputs

suppose $x(0_-) = 0$ and we apply input $u(t) = \delta^{(k)}(t)f$, where $\delta^{(k)}$ denotes $k$th derivative of $\delta$ and $f \in \mathbb{R}^m$

then $U(s) = s^k f$, so

$$X(s) = (sI - A)^{-1}Bs^k f = (s^{-1}I + s^{-2}A + \cdots) Bs^k f = \left( s^{k-1} + \cdots + sA^{k-2} + A^{k-1} + s^{-1}A^k + \cdots \right) B f$$

impulsive terms

hence

$$x(t) = \text{impulsive terms} + A^kBf + A^{k+1}Bf \frac{t}{1!} + A^{k+2}Bf \frac{t^2}{2!} + \cdots$$

in particular, $x(0+) = A^kBf$
thus, input $u = \delta^{(k)} f$ transfers state from $x(0-) = 0$ to $x(0+) = A^k B f$

now consider input of form

$$u(t) = \delta(t) f_0 + \cdots + \delta^{(n-1)}(t) f_{n-1}$$

where $f_i \in \mathbb{R}^m$

by linearity we have

$$x(0+) = B f_0 + \cdots + A^{n-1} B f_{n-1} = C \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

hence we can reach any point in $\text{range}(C)$

(at least, using impulse inputs)
can also be shown that any point in \( \text{range}(C) \) can be reached for any \( t > 0 \) using \textit{nonimpulsive} inputs

\textbf{fact}: if \( x(0) \in \mathcal{R} \), then \( x(t) \in \mathcal{R} \) for all \( t \) (no matter what \( u \) is)

to show this, need to show \( e^{tA}x(0) \in \mathcal{R} \) if \( x(0) \in \mathcal{R} \ldots \)
Example

- unit masses at $y_1$, $y_2$, connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^T \ y^T]^T$

system is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?
controllability matrix is

\[
C = \begin{bmatrix}
B & AB & A^2B & A^3B
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -2 & 2 \\
0 & -1 & 2 & -2 \\
1 & -2 & 2 & 0 \\
-1 & 2 & -2 & 0
\end{bmatrix}
\]

hence reachable set is

\[
\mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

we can reach states with \( y_1 = -y_2, \dot{y}_1 = -\dot{y}_2, \) \( i.e., \) precisely the differential motions

it’s obvious — internal force does not affect center of mass position or total momentum!
Least-norm input for reachability

(also called \textit{minimum energy input})

assume that $\dot{x} = Ax + Bu$ is reachable

we seek $u$ that steers $x(0) = 0$ to $x(t) = x_{\text{des}}$ and minimizes

$$\int_0^t \|u(\tau)\|^2 \, d\tau$$

let’s discretize system with interval $h = t/N$

(we’ll let $N \to \infty$ later)

thus $u$ is piecewise constant:

$$u(\tau) = u_d(k) \quad \text{for} \quad kh \leq \tau < (k + 1)h, \quad k = 0, \ldots, N - 1$$
so

\[ x(t) = \begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{N-1} B_d \end{bmatrix} \begin{bmatrix} u_d(N-1) \\ \vdots \\ u_d(0) \end{bmatrix} \]

where

\[ A_d = e^{hA}, \quad B_d = \int_0^h e^{\tau A} \, d\tau B \]

least-norm \( u_d \) that yields \( x(t) = x_{\text{des}} \) is

\[ u_{\text{dln}}(k) = B_d^T (A_d^T)^{(N-1-k)} \left( \sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i \right)^{-1} x_{\text{des}} \]

let’s express in terms of \( A \):

\[ B_d^T (A_d^T)^{(N-1-k)} = B_d^T e^{(t-\tau)A^T} \]
where $\tau = t(k + 1)/N$

for $N$ large, $B_d \approx (t/N)B$, so this is approximately

$$(t/N)B^Te^{(t-\tau)A^T}$$

similarly

$$\sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i = \sum_{i=0}^{N-1} e^{(ti/N)A}B_d B_d^T e^{(ti/N)A^T}$$

$$\approx (t/N) \int_{0}^{t} e^{\bar{t}A}BB^Te^{\bar{t}A^T} \, d\bar{t}$$

for large $N$
hence least-norm discretized input is approximately

\[ u_{ln}(\tau) = B^T e^{(t-\tau)A^T} \left( \int_0^t e^{\bar{t}A} BB^T e^{\bar{t}A^T} \, dt \right)^{-1} x_{des}, \quad 0 \leq \tau \leq t \]

for large \( N \)

hence, this is the least-norm continuous input

- can make \( t \) small, but get larger \( u \)
- cf. DT solution: sum becomes integral
min energy is
\[
\int_0^t \|u_{\text{ln}}(\tau)\|^2 d\tau = x_{\text{des}}^T Q(t)^{-1} x_{\text{des}}
\]
where
\[
Q(t) = \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau
\]
can show
\[
(A, B) \text{ controllable } \iff Q(t) > 0 \text{ for all } t > 0
\]
\[
\iff Q(s) > 0 \text{ for some } s > 0
\]
\[
in \text{ fact, range}(Q(t)) = \mathcal{R} \text{ for any } t > 0
\]
Minimum energy over infinite horizon

the matrix

\[ P = \lim_{t \to \infty} \left( \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau \right)^{-1} \]

always exists, and gives minimum energy required to reach a point \( x_{\text{des}} \) (with no limit on \( t \)):

\[
\min \left\{ \int_0^t \|u(\tau)\|^2 d\tau \ \bigg| \ x(0) = 0, \ x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}
\]

- if \( A \) is stable, \( P > 0 \) (i.e., can’t get anywhere for free)
- if \( A \) is not stable, then \( P \) can have nonzero nullspace
- \( P z = 0, \ z \neq 0 \) means can get to \( z \) using \( u \)'s with energy as small as you like (\( u \) just gives a little kick to the state; the instability carries it out to \( z \) efficiently)
General state transfer

consider state transfer from \( x(t_i) \) to \( x(t_f) = x_{\text{des}}, \ t_f > t_i \)

since

\[
x(t_f) = e^{(t_f-t_i)A}x(t_i) + \int_{t_i}^{t_f} e^{(t_f-\tau)A}Bu(\tau) \ d\tau
\]

\( u \) steers \( x(t_i) \) to \( x(t_f) = x_{\text{des}} \iff \)

\( u \) (shifted by \( t_i \)) steers \( x(0) = 0 \) to \( x(t_f - t_i) = x_{\text{des}} - e^{(t_f-t_i)A}x(t_i) \)

- general state transfer reduces to reachability problem

- if system is controllable, any state transfer can be effected
  - in ‘zero’ time with impulsive inputs
  - in any positive time with non-impulsive inputs
Example

- unit masses, springs, dampers
- $u_1$ is force between 1st & 2nd masses
- $u_2$ is force between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$
system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]
steer state from $x(0) = e_1$ to $x(t_f) = 0$

i.e., control initial state $e_1$ to zero at $t = t_f$

$\mathcal{E}_{\text{min}} = \int_0^{t_f} \|u_{\ln}(\tau)\|^2 d\tau$ vs. $t_f$:
for $t_f = 3$, $u = u_{\ln}$ is:
and for $t_f = 4$: 
output $y_1$ for $u = 0$: 
output $y_1$ for $u = u_{1n}$ with $t_f = 3$:
output $y_1$ for $u = u_{1n}$ with $t_f = 4$: 

![Graph showing the output $y_1$ for $u = u_{1n}$ with $t_f = 4$.](image)