

Basic Notation

Basic set notation

$\{a_1, \dots, a_r\}$	the set with elements a_1, \dots, a_r .
$a \in S$	a is in the set S .
$S = T$	the sets S and T are equal, <i>i.e.</i> , every element of S is in T and every element of T is in S .
$S \subseteq T$	the set S is a subset of the set T , <i>i.e.</i> , every element of S is also an element of T .
$\exists a \in S \mathcal{P}(a)$	there exists an a in S for which the property \mathcal{P} holds.
$\forall x \in S \mathcal{P}(x)$	property \mathcal{P} holds for every element in S .
$\{a \in S \mid \mathcal{P}(a)\}$	the set of all a in S for which \mathcal{P} holds (the set S is sometimes omitted if it can be determined from context).
$A \cup B$	union of sets, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
$A \cap B$	intersection of sets, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
$A \times B$	Cartesian product of two sets, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Some specific sets

\mathbf{R}	the set of real numbers.
\mathbf{R}^n	the set of real n -vectors ($n \times 1$ matrices).
$\mathbf{R}^{1 \times n}$	the set of real n -row-vectors ($1 \times n$ matrices).
$\mathbf{R}^{m \times n}$	the set of real $m \times n$ matrices.
j	can mean $\sqrt{-1}$, in the company of electrical engineers.
i	can mean $\sqrt{-1}$, for normal people; i is the polite term in mixed company (<i>i.e.</i> , when non-electrical engineers are present).
$\mathbf{C}, \mathbf{C}^n, \mathbf{C}^{m \times n}$	the set of complex numbers, complex n -vectors, complex $m \times n$ matrices.
\mathbf{Z}	the set of integers: $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$.
\mathbf{R}_+	the nonnegative real numbers, <i>i.e.</i> , $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$.
$[a, b], (a, b), [a, b), (a, b)$	the real intervals $\{x \mid a \leq x \leq b\}$, $\{x \mid a < x \leq b\}$, $\{x \mid a \leq x < b\}$, and $\{x \mid a < x < b\}$, respectively.

Vectors and matrices

We use square brackets [and] to construct matrices and vectors, with white space delineating the entries in a row, and a new line indicating a new row. For example [1 2] is a row vector in $\mathbf{R}^{1 \times 2}$, and $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is matrix in $\mathbf{R}^{2 \times 3}$. $[1 \ 2]^T$ denotes a column vector, *i.e.*, an element of $\mathbf{R}^{2 \times 1}$, which we abbreviate as \mathbf{R}^2 .

We use curved brackets (and) surrounding lists of entries, delineated by commas, as an alternative method to construct (column) vectors. Thus, we have three ways to denote a column vector:

$$(1, 2) = [1 \ 2]^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Note that in our notation scheme (which is fairly standard), [1, 2, 3] and (1 2 3) aren't used. We also use square and curved brackets to construct block matrices and vectors. For example if $x, y, z \in \mathbf{R}^n$, we have

$$\begin{bmatrix} x & y & z \end{bmatrix} \in \mathbf{R}^{n \times 3},$$

a matrix with columns x, y , and z . We can construct a block vector as

$$(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^{3n}.$$

Functions

The notation $f : A \rightarrow B$ means that f is a function on the set A into the set B . The notation $b = f(a)$ means b is the value of the function f at the point a , where $a \in A$ and $b \in B$. The set A is called the *domain* of the function f ; it can thought of as the set of legal parameter values that can be passed to the function f . The set B is called the *codomain* (or sometimes range) of the function f ; it can thought of as a set that contains all possible returned values of the function f .

There are several ways to think of a function. The formal definition is that f is a subset of $A \times B$, with the property that for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$. We denote this as $b = f(a)$.

Perhaps a better way to think of a function is as a *black box* or (software) *function* or *subroutine*. The domain is the set of all legal values (or data types or structures) that can be passed to f . The codomain of f gives the data type or data structure of the values returned by f .

Thus $f(a)$ is *meaningless* if $a \notin A$. If $a \in A$, then $b = f(a)$ is an element of B . Also note that the *function* is denoted f ; it is *wrong* to say 'the function $f(a)$ ' (since $f(a)$ is an element

of B , not a function). Having said that, we do sometimes use sloppy notation such as ‘the function $f(t) = t^3$ ’. To say this more clearly you could say ‘the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(t) = t^3$ for $t \in \mathbf{R}$ ’.

Examples

- $-0.1 \in \mathbf{R}$, $\sqrt{2} \in \mathbf{R}_+$, $1 - 2j \in \mathbf{C}$ (with $j = \sqrt{-1}$).
- The matrix

$$A = \begin{bmatrix} 0.3 & 6.1 & -0.12 \\ 7.2 & 0 & 0.01 \end{bmatrix}$$

is an element in $\mathbf{R}^{2 \times 3}$. We can define a function $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ as $f(x) = Ax$ for any $x \in \mathbf{R}^3$. If $x \in \mathbf{R}^3$, then $f(x)$ is a particular vector in \mathbf{R}^2 . We can say ‘the function f is linear’. To say ‘the function $f(x)$ is linear’ is technically wrong since $f(x)$ is a vector, not a function. Similarly we can’t say ‘ A is linear’; it is just a matrix.

- We can define a function $f : \{a \in \mathbf{R} \mid a \neq 0\} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ by $f(a, x) = (1/a)x$, for any $a \in \mathbf{R}$, $a \neq 0$, and any $x \in \mathbf{R}^n$. The function f could be informally described as division of a vector by a nonzero scalar.
- Consider the set $A = \{0, -1, 3.2\}$. The elements of A are 0, -1 and 3.2. Therefore, for example, $-1 \in A$ and $\{0, 3.2\} \subseteq A$. Also, we can say that $\forall x \in A$, $-1 \leq x \leq 4$ or $\exists x \in A$, $x > 3$.
- Suppose $A = \{1, -1\}$. Another representation for A is $A = \{x \in \mathbf{R} \mid x^2 = 1\}$.
- Suppose $A = \{1, -2, 0\}$ and $B = \{3, -2\}$. Then

$$A \cup B = \{1, -2, 0, 3\}, \quad A \cap B = \{-2\}.$$

- Suppose $A = \{1, -2, 0\}$ and $B = \{1, 3\}$. Then

$$A \times B = \{(1, 1), (1, 3), (-2, 1), (-2, 3), (0, 1), (0, 3)\}.$$

- $f : \mathbf{R} \rightarrow \mathbf{R}$ with $f(x) = x^2 - x$ defines a function from \mathbf{R} to \mathbf{R} while $u : \mathbf{R}_+ \rightarrow \mathbf{R}^2$ with

$$u(t) = \begin{bmatrix} t \cos t \\ 2e^{-t} \end{bmatrix}.$$

defines a function from \mathbf{R}_+ to \mathbf{R}^2 .