\[ \dot{x} = Ax \]

Find eigenvalues and eigenvectors for \( A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \)

Find solution to \( \det(\lambda I - A) = 0 \), which is when \( (\lambda + 1)(\lambda - 1) = 0 \)

\( \lambda_1 = 1, \lambda_2 = -1 \) and the corresponding eigenvectors are in the nullspace of \( \lambda I - A \) which is

\( v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

\( v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)
Vector Field

The vector field \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is \( \dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
Stable initial points in a generally unstable system

Note that for points along the direction [1 1] corresponding to the $\lambda_1 = 1$, the vector field points away from the origin since $\lambda_1 > 0$ and is unstable. And for points along the direction [0 1] corresponding to the $\lambda_2 = -1$, the vector field points towards the origin since $\lambda_1 < 0$. If the initial $x$ is exactly along [0 1] then it is stable.

In general, if

$$R\lambda_1 < 0, \cdots R\lambda_s < 0$$

$$R\lambda_{s+1} \geq 0, \cdots R\lambda_n \geq 0$$

What are the $x(0)$ such that the system is stable $x(t) \to 0$?

This is possible if $x(0) \in span[v_1, \cdots, v_s]$
Jordan Canonical form for solving discrete systems

- For any square matrix $A$, $A = TJT^{-1}$ where $J$ is made up of Jordan blocks along the diagonal $J = \begin{bmatrix} J_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & J_q \end{bmatrix}$, each Jordan block is of the form $J_i = \begin{bmatrix} \lambda_i & 1 & & \\
 & \lambda_i & \ddots & \\
 & \ddots & \ddots & 1 \\
 & & & \lambda_i \end{bmatrix}$.

- The number of times a particular $\lambda_i$ appears on the diagonal of $J$ is equal to the multiplicity of that $\lambda_i$.

- Thus for $\lambda_i$ with multiplicity 1, it is associated with a Jordan block of size 1.

- The discrete time solution to $x(t+1) = Ax(t)$ is $x(t) = A^t x(0) = TJ^t T^{-1} x(0)$. 
Jordan Canonical form for solving discrete systems

- One can see that when multiplied there will only be non-zero values within each Jordan block. So that $J^t = \begin{bmatrix} J^t_1 & \cdots & \cdots \cdots & \cdots \cdots & \cdots \cdots & \cdots \cdots & \cdots \cdots & J^t_q \end{bmatrix}$

- In particular for a 3x3 Jordan block 
  $$\begin{bmatrix} \lambda_i & 1 & \\ \lambda_i & 1 & \\ \lambda_i & 1 \end{bmatrix}^2 = \begin{bmatrix} \lambda_i^2 & 2\lambda_i & 1 \\ \lambda_i^2 & 2\lambda & \lambda_i^2 \end{bmatrix}$$

- And one can keep multiplying by the Jordan block to see that for a general t, $J^t_i = \begin{bmatrix} \lambda_i^t & f_{12}(\lambda_i) & f_{13}(\lambda_i) \\ \lambda_i^t & f_{23}(\lambda_i) & \lambda_i^t \end{bmatrix}$

  where $f_{12}(\lambda_i)$ and $f_{23}(\lambda_i)$ are polynomials of $\lambda_i$ of degree $t - 1$ and $f_{13}(\lambda_i)$ is a polynomial of degree $t - 2$.

- The generalized form of $x(t)$ is a sum where Jordan blocks of size $> 1$ contributes a complicated term with degree $\lambda_i^t$ while Jordan blocks of size 1 contribute the term $\lambda_i^t(w_i^T x(0))v_i$.
FIR filter

Example

Consider a cascade of 100 delays

\[ x(t + 1) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

where the states \( x \) is a vector of length 100 corresponding to each stage of the delay

\[ x(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_{100} \end{bmatrix} \]
Example

\[ x(t + 1) = \begin{bmatrix} 0 & 0 \\ I_{99} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} x(t) + 0 \cdot u(t) \]
FIR filter cont

Example

Write the linear dynamical system after adding a feedback term. The definition of the internal states $x$ remain unchanged, the equation for $y(t)$ is unchanged, the new difference equation is

$$x(t + 1) = \begin{bmatrix} 0 & \alpha \\ \mathbf{I}_{99} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$
Offset discretization

\[ \dot{x} = Ax + Bu \quad y = Cx + Du \]

We consider discretizing a continuous-time system where there is a constant time offset between the input update and output sampling. We define the sequences \( x_d \) and \( y_d \) as

\[ x_d(k) = x(kh) \quad y_d(k) = y(kh) \quad k = 0, 1, \ldots \]

where the state and output are sampled every \( h \) seconds. The input \( u \) is given by

\[ u(t) = u_d(k) \text{ for } kh + \delta \leq t < (k + 1)h + \delta \quad k = 0, 1, \ldots \]

where \( h > 0 \) and \( 0 \leq \delta < h \)
Offset discretization

Using \( x(t) = e^{tA}x(0) + \int_0^t e^{\tau A}Bu(t - \tau)d\tau \)

\( x_d(k+1) = x((k+1)h) = e^{hA}x(kh) + \int_0^h e^{\tau A}Bu((k+1)h - \tau)d\tau \)

\( = \cdots + \int_0^{h-\delta} e^{\tau A}Bu((k+1)h - \tau)d\tau + \int_{h-\delta}^h e^{\tau A}Bu((k+1)h - \tau)d\tau \)

\( = \cdots + \int_0^{h-\delta} e^{\tau A}Bd\tau u_d(k) + \int_{h-\delta}^h e^{\tau A}Bd\tau u_d(k-1) \)
Offset discretization

\[ x'_d(k + 1) = A_d x'_d(k) + B_d u_d(k) \quad y_d(k) = C_d x'_d(k) + D_d u_d(k) \]

We define the discrete internal state to be \( x'_d(k) = \begin{bmatrix} x_d(k) \\ u_d(k - 1) \end{bmatrix} \),

then \( A_d = \begin{bmatrix} e^{hA} & \int_{h-\delta}^h e^{\tau A} B_d \tau \\ 0 & 0 \end{bmatrix} \) and \( B_d = \begin{bmatrix} \int_{0}^{h-\delta} e^{\tau A} B_d \tau \\ 1 \end{bmatrix} \)

And \( C_d = \begin{bmatrix} C \\ 0 \end{bmatrix} \), \( D_d = D \)