1. **Linear algebra done right**

(a) Let $A \in \mathbb{R}^{n \times n}$, if $A^2 = A$ and rank($A$) = $n$, must $A$ be the identity matrix? Prove or disprove it with a counterexample.

(b) Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times p}$. Is the following always true:

$$\text{dim}(\text{range}(AB)) = \text{dim}(\text{range}(B)) - \text{dim}(-\text{null}(A) \cap \text{range}(B)),$$

where $\text{dim}(\mathcal{V})$ gives the dimension of the vector space $\mathcal{V}$. Prove or disprove it with a counterexample.

**Note:** The operation $\cap$ denotes intersection. That is, given any two sets $A$ and $B$, $A \cap B$ is the set that contains elements both in $A$ and in $B$. Consequently, $\text{null}(A) \cap \text{range}(B)$ in the problem means the set of all vectors that are in both the null space of $A$ and the range of $B$.

(c) Let $u_1, \ldots, u_k \in \mathbb{R}^m$ and $v_1, \ldots, v_k \in \mathbb{R}^n$ be column vectors. Consider the matrix $M = \sum_{i=1}^{k} u_i v_i^T$. Compare the rank($M$) with $k$ and prove your claim.

(d) Let $M \in \mathbb{R}^{m \times n}$ be a rank $k$ matrix. Is it always possible to find column vectors $u_1, \ldots, u_k \in \mathbb{R}^m$ and $v_1, \ldots, v_k \in \mathbb{R}^n$ such that $M = \sum_{i=1}^{k} u_i v_i^T$? Prove or disprove (with a counter example) the claim.

**Note:** You should only use material covered in EE263 lectures up till this point.

(e) Let $G = (V,E)$ be a simple and undirected graph, where $V = \{1, 2, \ldots, n\}$ is the set of $n$ vertices and $E$ is the set of all edges. Let the matrix $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of this graph $G$, defined as follows:

$$A_{ij} = \begin{cases} 
1, & (i, j) \text{ or } (j, i) \in E \\
0, & \text{otherwise}
\end{cases} \quad (1)$$

Note that since the graph $G$ being simple means that there is no self-loop, hence $A_{ii} = 0$ for each $i$. Further, since the graph is undirected, $A_{ij} = A_{ji}, \forall i, j \in V$.

A triangle in $G$ is a set of three distinct vertices $i, j, k$ where $(i, j) \in E, (j, k) \in E, (k, i) \in E$. Let $T$ be the total number of triangles in the graph $G$. Prove the following statement:

$$\text{Tr}(A^3) = 6T,$$

where $\text{Tr}(A)$ gives the trace of the square matrix $A \in \mathbb{R}^{n \times n}$, which is equal to the sum of all its diagonal entries (i.e. $\text{Tr}(A) = \sum_{i=1}^{n} A_{ii}$).

**Note:** To help you better understand the problem, consider the below example graph:
where the vertex set is $V = \{1, 2, 3, 4\}$, and the edge set is $E = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4)\}$. The adjacency matrix is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

You can easily check that in this case the number of triangles is 2 and $\text{Tr}(A^3) = 12$. As a final hint, in Review Session 4, we discussed what each entry in $A^2$ means. Think about what each entry on the diagonal of $A^3$ gives you.

**Solution.**

(a) $A$ must be the identity matrix, because $\text{rank}(A) = n$ implies that $A$ is invertible. Consequently, let $A^{-1}$ be $A$’s inverse, and multiply $A^{-1}$ on both sides of the equation $A^2 = A$ yields the result.

(b) True. Consider the linear transformation $T : \text{range}(B) \rightarrow \text{range}(AB)$ given by

$$T(x) = Ax, \quad \forall x \in \text{range}(B)$$

Note that $T(x) = 0$ if and only if $x \in \text{range}(B)$ and $Ax = 0$. Consequently, the dimension of the nullspace (kernel) of $T$ is $\text{dim}(\text{null}(A) \cap \text{range}(B))$. By rank-nullity theorem, we have

$$\text{dim}(\text{range}(AB)) + \text{dim}(\text{null}(A) \cap \text{range}(B)) = \text{dim}(\text{range}(B)),$$

which yields the result.

(c) $\text{rank}(M) \leq k$. Note that $Mx = \sum_{i=1}^{k} u_i v_i^T x$, which is a linear combination of $k$ vectors $u_1, \ldots, u_k$. Consequently, the span has dimension at most $k$.

(d) It is always possible. Since $M$ has rank $k$, pick $k$ linearly independent columns (out of $n$ columns in total) that span the column space of $M$: call them $u_1, \ldots, u_k$. 


Consequently, each column in \( M \) can be written as a linear combination of those \( k \) columns, which leads to
\[
M = UV,
\]
where \( U = [u_1, u_2, \ldots, u_k] \) and each column \( \bar{v}_i \) (\( i = 1, 2, \ldots, n \)) of \( V \) is the linear combination coefficient vector. Namely, for each column \( c_i \) of \( M \), we have \( c_i = U\bar{v}_i \).

Denote each row of \( V \) to be \( v_i^T \), we therefore have \( M = \sum_{i=1}^k u_i v_i^T \).

(e) First, by expanding the trace, we have
\[
\text{Tr}(A^3) = \sum_i \sum_j \sum_k A_{ij} A_{jk} A_{ki}
\]
If \( i = j \) or \( j = k \) or \( k = i \), then \( A_{ij} A_{jk} A_{ki} = 0 \). Consequently, the only case where \( A_{ij} A_{jk} A_{ki} \) is not 0 is when \( i \neq j \) and \( j \neq k \) and \( k \neq i \) and \( A_{ij} = 1, A_{jk} = 1, A_{ki} = 1 \), in which case \( i, j, k \) form a triangle. However, for each such triangle \((i, j, k)\), there are six permutations of \((i, j, k)\) that contributes to the sum. Consequently, one needs to multiply the total number of triangles by 6.
3. Coin collector robot. Consider a robot with unit mass which can move in a frictionless two dimensional plane. The robot has a constant unit speed in the $y$ direction (towards north), and it is designed such that we can only apply force in the $x$ direction. We will apply a force at time $t$ given by $f_j$ for $2j - 2 \leq t < 2j$ where $j = 1, \ldots, n$, so that the applied force is constant over time intervals of length 2. The robot is at the origin at time $t = 0$ with zero velocity in the $x$ direction.

There are $2n$ coins in the plane and the goal is to design a sequence of input forces for the robot to collect the maximum possible number of coins. The robot is designed such that it can collect the $i$th coin only if it exactly passes through the location of the coin $(x_i, y_i)$. In this problem, we assume that $y_i = i$.

(a) Find the coordinates of the robot at time $t$, where $t$ is a positive integer. Your answer should be a function of $t$ and the vector of input forces $f \in \mathbb{R}^n$.

(b) Given a sequence of $k$ coins $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$, describe a method to find whether the robot can collect them.

(c) For the data provided in robot_coin_collector.m, show that the robot cannot collect all the coins.

(d) Suppose that there is an arrangement of the coins such that it is not possible for the robot to collect all the coins. Suggest a way to check if the robot can collect all but one of the coins.

(e) Run your method on data in robot_coin_collector.m and report which coin cannot be collected. Report the input that results in collecting $2n - 1$ coins. Plot the location of the coins and the location of the robot at integer times.

Solution.

(a) The second coordinate at time $t$ is simply equal to $t$.

Consider $A \in \mathbb{R}^{2n \times n}$ such that

$$A_{ij} = \begin{cases} 1 & j = \left\lfloor \frac{i+1}{2} \right\rfloor \\ 0 & \text{Otherwise} \end{cases}$$

Then we will have $Af = [f_1, f_1, f_2, \ldots, f_n]$. Similar to the mass/force example, the first coordinate at time $t$ will be equal to $b_t^T Af$ where

$$b_t = [t - \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0]^T.$$  

(b) According to part a, the only possible time to collect the $i$th coin is at time $t = y_i = i$. Define $l_i$ to be the first coordinate of the location of the robot at time $t = i$. From part a, we see that

$$l_i = b_i^T Af.$$ 

Let $B \in \mathbb{R}^{n \times n}$ be a matrix whose $i$th column is $b_i$ and define $C = B^T A$. Then we will have $l = Cf$.

Hence, we see that the necessary and sufficient condition to collect all the coins is that $x \in \mathcal{R}(C)$. This can be simply examined with $\text{rank}([C \ x]) = \text{rank}(C)$. 

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(c) The code to solve parts c,e can be find at the bottom.

(d) In part b, we saw that $l = Cf$. We know that there exists a sequence of input forces $f$ such that all but one of the $2n$ equations are satisfied, but we don’t know which one.

Let $x^{(i)}$ be the location vector $x$ with the $i$th entry removed. Likewise, let $C^{(i)}$ be the transition matrix with the $i$th row of $C$ removed. If we can collect all coins but the $i$th one, then we will certainly have $x^{(i)} \in \mathcal{R}(C^{(i)})$. We will loop over the coins and see whether it’s possible to collect all coins but one.

(e) The following code solves the problem:

```matlab
clc
clear all
close all
robot_coin_collector

BT = zeros(2*n,2*n);
for i=1:2*n
    BT(i,1:i) = i-1/2:-1:1/2;
end

A = zeros(2*n,n);
for i=1:n
    A([2*i-1,2*i],i)=1;
end
C = BT*A;

%part c
if rank([C,x])==rank(C)
    fprintf('All coins can be collected!
')
else
    fprintf('All coins cannot be collected!
')
end

%part e
for i=1:2*n
    xt = x([1:i-1,i+1:end]);
    Ct = C([1:i-1,i+1:end],:);
    if rank([Ct,xt])==rank(Ct)
        fprintf('The robot can collect all coins but %dth,
',i);
        fprintf('and the input will be: \n')
        input = Ct\xt;
        disp(input)
    end
end
```

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Figure 1: Location of the coins and the trajectory of the robot

We see that all coins but the 7th can be collected and the associated input will be

\[ f = [1.0000, -4.0000, 7.0000, -10.0000, 20.0000, -35.0000]. \]