1. Finding a basis for the intersection of ranges.

a) Suppose you are given two matrices, $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{n \times q}$. Explain how you can find a matrix $C \in \mathbb{R}^{n \times r}$, with independent columns, for which

$$\text{range}(C) = \text{range}(A) \cap \text{range}(B).$$

This means that the columns of $C$ are a basis for $\text{range}(A) \cap \text{range}(B)$.

b) Carry out the method described in part (a) for the particular matrices $A$ and $B$ defined in intersect_range_data.m. Be sure to give us your matrix $C$, as well as the matlab (or other) code that generated it. Verify that $\text{range}(C) \subseteq \text{range}(A)$ and $\text{range}(C) \subseteq \text{range}(B)$, by showing that each column of $C$ is in the range of $A$, and also in the range of $B$.

Please carefully separate your answers to part (a) (the general case) and part (b) (the specific case).

Solution.

a) We know that

$$\text{range}(A) = \text{null}(A^T)^\perp.$$

This means that any $y$ in $\text{range}(A)$ is perpendicular to all vectors in the $\text{null}(A^T)$; and any vector which is perpendicular to all vectors in $\text{null}(A^T)$, must be in $\text{range}(A)$. We will show that

$$\text{range}(A) \cap \text{range}(B) = \left(\text{null}(A^T) + \text{null}(B^T)\right)^\perp.$$

Let $y$ be a vector in $\text{range}(A) \cap \text{range}(B)$. Then $y = Ax_a$, for some $x_a$ and $y = Bx_b$, for some $x_b$. Let $v$ be any vector in $\text{null}(A^T) + \text{null}(B^T)$. Then $v = v_a + v_b$ for some $v_a \in \text{null}(B^T), v_b \in \text{null}(B^T)$. Then we have,

$$y^Tv = y^Tv_a + y^Tv_b = x_a^TA^Tv_a + x_b^TB^Tv_b = x_a^T(A^Tv_a) + x_b^T(B^Tv_b) = 0.$$

Thus $y \perp (\text{null}(A^T) + \text{null}(B^T))$. Since any vector in $(\text{range}(A) \cap \text{range}(B))$ is perpendicular to every vector in $(\text{null}(A^T) + \text{null}(B^T))$,

$$\text{range}(A) \cap \text{range}(B) \subseteq \left(\text{null}(A^T) + \text{null}(B^T)\right)^\perp.$$
Let $y$ be a vector in $(\text{null}(A^T) + \text{null}(B^T))^\perp$. Then $y$ is perpendicular to all vectors in $\text{null}(A^T)$ which means $y \in \text{range}(A)$. Similarly $y$ is perpendicular to all vectors in $\text{null}(B^T)$ which means $y \in \text{range}(B)$. Thus $y \in (\text{range}(A) \cap \text{range}(B))$ and we have,

\[
\text{range}(A) \cap \text{range}(B) = \left(\text{null}(A^T) + \text{null}(B^T)\right)^\perp.
\]

The full QR factorization of a matrix $A$ is,

\[
A = [Q_1 A \ Q_2 A] \begin{bmatrix} R_{1A} & 0 \\ 0 & 0 \end{bmatrix},
\]

and $\text{null}(A^T) = \text{range}(Q_{2A})$. Similarly, let full QR factorization if a matrix $B$ be

\[
B = [Q_1 B \ Q_2 B] \begin{bmatrix} R_{1B} & 0 \\ 0 & 0 \end{bmatrix},
\]

and hence $\text{null}(B^T) = \text{range}(Q_{2B})$. Then,

\[
\text{null}(A^T) + \text{null}(B^T) = \text{range}(Q_{2A}) + \text{range}(Q_{2B}) = \text{range}(D),
\]

where $D = [Q_{2A} \ Q_{2B}]$. Now,

\[
\text{range}(A) \cap \text{range}(B) = \left(\text{null}(A^T) + \text{null}(B^T)\right)^\perp = \text{range}(D)^\perp = \text{null}(D^T).
\]

So we find the QR factorization of $D$. Let the QR factorization be

\[
D = [Q_1 D \ Q_2 D] \begin{bmatrix} R_{1D} & 0 \\ 0 & 0 \end{bmatrix},
\]

and then $C = Q_{2D}$ as $\text{null}(D^T) = \text{range}(Q_{2D}) = \text{range}(C)$. Thus we have the matrix $C$ such that $\text{range}(C) = \text{range}(A) \cap \text{range}(B)$.

b) The following matlab code gives the required matrix $C$ and the dimension of $\text{range}(C)$.

```matlab
% clear commands
% intersect_range_data; % load data
Q_2A = null(A');
Q_2B = null(B');
D = [Q_2A Q_2B];
C = null(D');
rC = rank(C)
>>
C =
-0.3365  -0.2349   0.3581
 0.2927  -0.4471  -0.0277
-0.6691   0.0460  -0.1406
0.1963   0.3655  -0.2581
 0.3599  -0.1406  -0.1416
```

2
\[-0.0929 \quad 0.1880 \quad -0.5108 \\
0.1967 \quad 0.4497 \quad 0.3712 \\
0.2019 \quad -0.5007 \quad 0.0800 \\
0.2901 \quad 0.2292 \quad 0.2283 \\
-0.1140 \quad -0.2208 \quad 0.5718 \]

\[rC = 3\]

Show \(\text{range}(C) \subseteq\text{range}(A)\) and \(\text{range}(C) \subseteq\text{range}(B)\).

\[rA = \text{rank}(A)\]
\[rAC = \text{rank}(\begin{bmatrix} A & C \end{bmatrix})\]
\[rB = \text{rank}(B)\]
\[rBC = \text{rank}(\begin{bmatrix} B & C \end{bmatrix})\]

\[>>\]
\[rA = 6\]
\[rAC = 6\]
\[rB = 5\]
\[rBC = 5\]