EE263: Introduction to Linear Dynamical Systems

Review Session 2

Basic concepts from linear algebra

- matrix equations and interpretations
- vector spaces
- nullspace
Node adjacency matrix

- graph with $n$ nodes and (undirected) edges

- node adjacency matrix $A \in \mathbb{R}^{n \times n}$ is given by

\[
A_{ij} = \begin{cases} 
1 & \text{there is an edge connecting node } i \text{ and node } j \\
0 & \text{otherwise}
\end{cases}
\]

- what is meaning of $(i, j)$th entry of $A^k$?
Node adjacency matrix (cont...) 

Solution.

- Let’s start with $k = 2$

- $A_{ik}A_{kj} = 1$ only if there is an edge from node $i$ to node $k$, and an edge from node $k$ to node $j$, i.e., there is a path of length 2 between $i$ and $j$, passing through node $k$

- $(A^2)_{ij} = \sum_{k=1}^{n} A_{ik}A_{kj}$ which is the number of paths of length 2 between $i$ and $j$

- In general case, $(A^k)_{ij}$ is the number of paths of length $k$ between nodes $i$ and $j$
Node incidence matrix

- graph with \( n \) nodes and \( m \) directed edges

- node incidence matrix \( A \in \mathbb{R}^{n \times m} \) is defined as

\[
A_{i,j} = \begin{cases} 
1 & \text{edge } j \text{ enters (points into) node } i \\
-1 & \text{edge } j \text{ leaves (points out of) node } i \\
0 & \text{otherwise.}
\end{cases}
\]

- what does \( y = Ax \) mean?

- what does \( z = A^T w \) mean?

- what is \( AA^T \)?
Node incidence matrix (contd…)

Solution.

- defining $x_j$ to be flow rate on edge $j$, $y_i$ is the total flow into node $i$ (out of, if negative)

- for $w_j \in \{0, 1\}$, $z_i$ is the sum rate on edge $i$

- $(AA^T)_{ii} =$ number of edges connected to the $i$th node

- for $i \neq j$, $(AA^T)_{ij} = -1$, only if there is an edge from node $i$ to node $j$
Prerequisites

We assume that you are familiar with the basic definitions of the following concepts from lecture 3:

- vector spaces
- subspaces
- independence
- span
- basis
- dimension
Nullspace of a matrix

- For a matrix $A \in \mathbb{R}^{m \times n}$, the nullspace is defined as,
  \[
  \text{null}(A) = \{ x \in \mathbb{R}^n | Ax = 0 \}.
  \]

- Is $\text{null}(A) \subseteq \mathbb{R}^N$ a vector subspace of $\mathbb{R}^n$? Can you prove it?
  
  *Solution:* take two vectors $v_1, v_2 \in \text{null}(A)$, and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$. Then,
  \[
  A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 Av_1 + \alpha_2 Av_2 = 0.
  \]
  So $\alpha_1 v_1 + \alpha_2 v_2 \in \text{null}(A)$.

- Roughly speaking, to verify that a set $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace, we need only check that it is closed under vector addition and scalar multiplication.
Example 1

Let

\[
P = \begin{bmatrix}
A \\
A + B \\
A + B + C
\end{bmatrix}.
\]

▶ True or false?

\[\text{null}(P) = \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)\]

▶ Note that \(\text{null}(P)\) is a set, and \(\text{null}(A) \cap \text{null}(B) \cap \text{null}(C)\) is also a set.

▶ We say that two sets \(X\) and \(Y\) are equal if \(z \in X \Rightarrow z \in Y\), and \(z \in Y \Rightarrow z \in X\).

Solution:

▶ syntax check: Two subspaces can only be equal if they contain vectors of the same size:

LHS: if \(A \in \mathbb{R}^{m \times n}\), then \(B \in \mathbb{R}^{m \times n}\) and \(C \in \mathbb{R}^{m \times n}\). Therefore, \(P\) has \(n\) columns, and so \(\text{null}(P)\) is a subspace of \(\mathbb{R}^n\).

RHS: \(\text{null}(A)\), \(\text{null}(B)\), and \(\text{null}(C)\) are all subspaces of \(\mathbb{R}^n\), and hence their intersection must also be a subspace of \(\mathbb{R}^n\).
show that $x \in \text{null}(P) \Rightarrow \text{null}(A) \cap \text{null}(B) \cap \text{null}(C)$.

Let $x \in \text{null}(P)$. This implies

\[
\begin{align*}
Ax &= 0 \\
Ax + Bx &= 0 \\
Ax + Bx + Cx &= 0
\end{align*}
\]

\[
\begin{align*}
x \in \text{null}(A) \\
x \in \text{null}(B) \\
x \in \text{null}(B) \\
x \in \text{null}(B)
\end{align*}
\]

show that $x \in \text{null}(A) \cap \text{null}(B) \cap \text{null}(C) \Rightarrow x \in \text{null}(P)$

This is trivial: if $Ax = 0$, $Bx = 0$, $Cx = 0$ then $Ax + Bx = 0$ and $Ax + Bx + Cx = 0$, so $x \in \text{null}(P)$. 
Example 2

Is this true or false?

\[ \text{null}(A^T A) = \text{null}(A). \]

Solution:

▸ syntax check: If \( A \in \mathbb{R}^{m \times n} \) then \( A^T A \in \mathbb{R}^{n \times n} \), so \( \text{null}(A^T A) \) and \( \text{null}(A) \) are both subspaces of \( \mathbb{R}^n \).

▸ show that \( x \in \text{null}(A) \Rightarrow x \in \text{null}(A^T A) \):

\[
Ax = 0 \Rightarrow (A^T A)x = A^T(Ax) = A^T 0 = 0,
\]

so \( x \in \text{null}(A^T A) \).

▸ show that \( x \in \text{null}(A^T A) \Rightarrow x \in \text{null}(A) \):

Suppose \( x \in \text{null}(A^T A) \). Then, \( A^T Ax = 0 \) and so

\[
x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 = 0 \Rightarrow \|Ax\| = 0.
\]

▸ norm of a vector is zero if and only if the vector is equal to zero, \( i.e., \)

\[
\|z\| = 0 \Leftrightarrow z = 0.
\]

▸ \( \|Ax\| = 0 \) therefore implies that \( Ax = 0 \), and so \( x \in \text{null}(A) \).