0.1 Importing Variables from a MATLAB .m file

If you are importing variables given in file vars.m, use the following code at the beginning of your script.

\[
\text{close all; clearvars; vars;}
\]

1.1 Matrix-Vector Multiplication Review

In class, we saw two ways we can view the matrix-vector multiplication \( y = Ax, A \in \mathbb{R}^{m \times n} \).

1. Columns: \( y \) is a linear combination of the columns of \( A \). Consider the columns of \( A, a_j \) for \( j \in \{1, 2, \ldots, n\} \):

\[
A = \begin{bmatrix}
\vdots & \vdots & \hdots & \vdots \\
a_1 & a_2 & \ldots & a_n
\end{bmatrix}
\]

The vector \( y \) is a linear combination of \( A \)'s columns \( a_j \) with coefficients \( x_j \), the jth component of \( x \):

\[
y = Ax = A \left( \sum_{j=1}^{n} x_j e_j \right) = \sum_{j=1}^{n} x_j A e_j = \sum_{j=1}^{n} x_j a_j \quad \implies \quad y = \sum_{j=1}^{n} x_j a_j
\]

2. Rows: Each component of \( y \) is the inner product a row of \( A \) with \( x \). Consider the rows of \( A, \tilde{a}_i^T \) for \( i \in \{1, 2, \ldots, m\} \):

\[
A = \begin{bmatrix}
- \tilde{a}_1^T & - \\
- \tilde{a}_2^T & - \\
\vdots & \vdots \\
- \tilde{a}_m^T & -
\end{bmatrix}
\]

Each entry of \( y \) is the inner product of the corresponding row of \( A \) with \( x \), defining a hyperplane:

\[
y = Ax \implies y_i = \tilde{a}_i^T x
\]
1.2 Nullspace Review

We defined the nullspace of matrix $A \in \mathbb{R}^{m \times n}$, denoted by $N(A)$ as follows:

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

Note that the nullspace of a matrix, $N(A)$, happens to be a vector space (and a subspace of $\mathbb{R}^n$). This means it has to satisfy a few properties:

1. The zero vector $0 \in N(A)$
2. Closure under addition
3. Closure under scalar multiplication

The nullspace includes $0$ since $A0 = 0$. We can show the second two properties together. Consider $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v_1, v_2 \in N(A)$. We must show that $\alpha_1 v_1 + \alpha_2 v_2 \in N(A)$:

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 Av_1 + \alpha_2 Av_2 = \alpha_1 0 + \alpha_2 0 = 0$$

Thus, the nullspace $N(A)$ is a vector space and subspace of $\mathbb{R}^n$.

What if $N(A) = \{0\}$? All of the following statements are equivalent to each other. It’s helpful to reason through each of them.

1. $N(A) = \{0\}$
2. $x$ is uniquely determined from $y = Ax$
3. The mapping from $x$ to $y = Ax$ is one-to-one (different $x$’s map to different $y$’s)
4. The columns of $A$ are independent
5. $A$ has a left inverse, i.e., $\exists$ matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
6. $A^TA$ is invertible

2.1 Ex 1: Finding an unknown point on a sphere with rangings

Consider a unit sphere $S^n \in \mathbb{R}^n$:

$$S^n = \{ x \in \mathbb{R}^n \mid \|x\|_2 = 1 \}$$

We want to place $k$ beacons on the sphere $p_1, p_2, \ldots, p_k \in S^n$ such that given the distance along the sphere from $p_i$ to an unknown point $x \in S^n$ for all $i = 1, 2, \ldots, k$, we can uniquely determine $x$.

More concretely, you’re given $d_i = \text{sphdist}(x, p_i) = \text{for } i \in \{1, 2, \ldots, k\}$. Since this is a unit sphere, the distance along the sphere (arc length) is the same as the angle between $x$ and $p_i$, $\angle(x, p_i)$. We assume we always are given the shortest distance so that $\text{sphdist}(x, p_i) \in [0, \pi]$. We want to find conditions on the $p_i$’s such that we can always unambiguously determine $x \in S^n$ from the $d_i$’s.
Solution:

We want to turn the spherical distance into something easier to work with. We note that there is a relationship between $d_i = \angle(x, p_i)$ and $p_i^T x$. Specifically,

$$p_i^T x = \|p_i\|\|x\| \cos d_i = \cos d_i.$$  

Since $\|p_i\| = \|x\| = 1$, as these points lie on $S^n$, we can arrange these $k$ equations in a matrix format.

$$y = \begin{bmatrix} \cos d_1 \\ \cos d_2 \\ \vdots \\ \cos d_k \end{bmatrix} = \begin{bmatrix} - \tilde{p}_1^T \\ - \tilde{p}_2^T \\ \vdots \\ - \tilde{p}_k^T \end{bmatrix} x = Ax$$

We want the equation above to have exactly one solution so that we can determine the point $x$ unambiguously. This is equivalent to needing $\mathcal{N}(A) = \{0\}$.

### 3.2 Matrix-Matrix Multiplication

In class, we saw four ways we can view the matrix-matrix multiplication $C = AB$, where $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, which tells us that $C \in \mathbb{R}^{m \times p}$.

1. **Element-wise:** $c_{i,j}$ is the inner product of a row of $A$ with a column of $B$. Consider the rows of $A$, $\tilde{a}_i^T$ for $i \in \{1, 2, \ldots, m\}$ and the columns of $B$, $b_j$ for $j \in \{1, 2, \ldots, p\}$

   $$C = AB = \begin{bmatrix} - \tilde{a}_1^T \\ \vdots \\ - \tilde{a}_m^T \end{bmatrix} \begin{bmatrix} b_1 & \ldots & b_p \end{bmatrix} \Rightarrow c_{i,j} = \tilde{a}_i^T b_j = \sum_{k=1}^n a_{i,k} b_{k,j}$$

2. **Columns of $B$:** Each column of $C$ is $A$ times a column of $B$.

   $$C = AB = A \begin{bmatrix} b_1 & \ldots & b_p \end{bmatrix} = \begin{bmatrix} A b_1 & \ldots & A b_p \end{bmatrix}$$

3. **Rows of $A$:** Each row of $C$ is a row of $A$ multiplied by $B$.

   $$C = AB = \begin{bmatrix} - \tilde{a}_1^T \\ \vdots \\ - \tilde{a}_m^T \end{bmatrix} B = \begin{bmatrix} - \tilde{a}_1^T B \\ \vdots \\ - \tilde{a}_m^T B \end{bmatrix}$$

4 **Outer Products:** of each column of $A$ and row of $B$.

   $$C = AB = \sum_{k=1}^n a_{i,k} \tilde{b}_k^T$$
4.1 Ex 2. Graph Adjacency Matrix

We define the Adjacency Matrix $A$ for some non-directional graph with $n$ nodes as

$$a_{i,j} = \begin{cases} 
1 & \text{if node } i \text{ is connected to node } j \\
0 & \text{otherwise}
\end{cases}$$

For example, consider the graph below. We can determine its adjacency matrix $A$.

\[
\begin{array}{c|ccc}
\text{nodes} & 1 & 2 & 3 \\
nodes & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
3 & 0 & 1 & 0 \\
\end{array}
\]

What is $A^r$ for some adjacency matrix $A$ and some positive integer $r$?

**Solution:** Let’s start with the case $k = 2$. We have

$$[A^2]_{i,j} = \sum_{k=1}^{n} a_{i,k}a_{k,j}$$

We can examine each of the summands $a_{i,k}a_{k,j}$. Note that each element of $A$ can only take on the values 0 or 1, so the product equals 1 iff each multiplicand equals 1, which is only the case when both $i$ & $k$ and $k$ & $j$ are connected

$$a_{i,k}a_{k,j} = \begin{cases} 
1 & \text{if node } i \text{ is connected to node } j \text{ through node } k \\
0 & \text{otherwise}
\end{cases}$$

We sum this quantity for all $n$ nodes in the graph, thus

$$\sum_{k=1}^{n} a_{i,k}a_{k,j} = \text{the number of paths from } i \text{ to } j \text{ through another node}$$

In other words, $[A^2]_{i,j}$ is the number of paths of length 2 from $i$ to $j$, or equivalently the number of other nodes connected directly to both $i$ and $j$. In the general case $[A^r]_{i,j}$ is the number of paths of length $r$. We could prove this using an inductive argument.

Aside: This might look similar to HW1, Q8. There, we worked with a directed graph, but the problem could be formulated as counting the a number of ways to traverse a graph that had certain properties. Graphs can come up in all sorts of applications, from modeling interactions in social networks to routing traffic through various types of transportation systems.
4.2 Ex 3. Incidence Matrices in Directed Graphs

Consider some directed graph (i.e., the edges are one-way, instead of two-way) that has \( n \) nodes and \( m \) edges. Consider the incidence matrix (or arc-node incidence matrix) \( A \in \mathbb{R}^{n \times m} \)

\[
a_{i,j} = \begin{cases} 
+1 & \text{if edge } j \text{ enters node } i \\
-1 & \text{if edge } j \text{ exits node } i \\
0 & \text{otherwise}
\end{cases}
\]

Assume you’re given \( x \in \mathbb{R}^m \) such that \( x_j \) is the amount of flow in edge \( j \). What does the product \( Ax \) represent?

**Solution:** Define \( y = Ax \). First we note that \( y \in \mathbb{R}^n \), so it must say something about each of the nodes. Consider one of the entries \( y_i \). We can write this as

\[
y_i = \tilde{a}_i^T x = \sum_{j=1}^m a_{i,j} x_j
\]

We know that \( a_{i,j} \) only takes values in \( \{0, \pm 1\} \). Working through the cases, we have

\[
\sum_{j=1}^m a_{i,j} x_j = \begin{cases} 
+x_j & \text{if edge } j \text{ enters node } i \\
-x_j & \text{if edge } j \text{ exits node } i \\
0 & \text{otherwise}
\end{cases}
\]

We conclude that \( y_i \) is the net amount of flow into node \( i \).

4.2.1 What does the matrix \( M = AA^T \) represent?

**Solution:** First, a quick dimensions check tells us that \( M \in \mathbb{R}^{n \times n} \), so we expect this matrix to tell us something about relationships between the nodes. We can split this product into two parts: the entries on the diagonal of \( M \), and the entries off the diagonal.

\[
M = AA^T = \begin{bmatrix}
-\tilde{a}_1^T & \vdots & -\tilde{a}_n^T
\end{bmatrix} \begin{bmatrix}
| & | & | \\
\tilde{a}_1 & \ldots & \tilde{a}_n
\end{bmatrix} \implies m_{i,i} = \tilde{a}_i^T \tilde{a}_i = \sum_{j=1}^m a_{i,j}^2
\]

Going back to the definition of the incidence matrix, we see that \( a_{i,j}^2 \) equals 1 if node \( j \) either enters or exits node \( i \), and 0 otherwise. We sum this over all edges, thus

\[
m_{i,i} = \text{the number of edges connected to node } i \text{ (also called the degree of node } i)\]

Similarly we look at the entries off the diagonal and examine the summands.

\[
m_{i,j} = \tilde{a}_i^T a_j = \sum_{k=1}^m a_{i,k} a_{j,k}
\]
\[ a_{i,k}a_{j,k} = \begin{cases} -1 & \text{if edge } k \text{ enters node } i \text{ and exits node } j, \text{ or vice versa} \\ 0 & \text{otherwise} \end{cases} \]

We sum this over all the edges, so we will see if \( i \) and \( j \) are connected by one of them.

\[ m_{i,j} = \begin{cases} -1 & \text{if nodes } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases} \]

Note that I assumed there exists only one edge between nodes (which has some direction). However, if you consider multiple edges between each node (multigraph), or the possibility of both directed and non-directed edges (which we could view as two edges between the nodes, one in each direction), the definition of \( m_{i,j} \) would change slightly. In fact in the formulation above, a single edge \( k \) cannot both enter and exit a node \( i \), as that would require \( a_{i,k} \) to be \( \pm 1 \) simultaneously. This means there are no self-loops (edges from \( i \) to \( i \)) or bidirectional edges.